

# Self-Fulfilling Risk Panics: technical note

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February 6, 2012

## **Abstract**

# 1 Switching Equilibria with Markov

Denote the low risk state as state 1 (like the fundamental equilibrium) and the high risk state as state 2 (like the sunspot-like equilibrium). The asset price can then take on the values  $Q_N(1)$ ,  $Q_N(2)$ ,  $Q_B(1)$  and  $Q_B(2)$ . Let  $p_1$  be the probability of staying in state 1 if we are currently in state 1 and  $p_2$  the probability of staying in state 2 if we are currently in state 2.

The asset market clearing is

$$E_t (A_{t+1} + Q_{t+1} - RQ_t) = \frac{\gamma K}{W} \text{var}_t (Q_{t+1} + A_{t+1})$$

## 1.1 General expressions

Consider that the probability to be in state 1,  $N$  is  $\rho_{1N}$ , in state 1,  $B$  is  $\rho_{1B}$ , in state 2,  $N$  is  $\rho_{2N}$ , in state 2,  $B$  is  $\rho_{2B}$ . Then

$$\begin{aligned} & E_t (Q_{t+1} + A_{t+1}) \\ &= \rho_{1N} (Q_N(1) + A_N) + \rho_{1B} (Q_B(1) + A_B) \\ & \quad + \rho_{2N} (Q_N(2) + A_N) + \rho_{2B} (Q_B(2) + A_B) \end{aligned} \tag{1}$$

and

$$\begin{aligned} & E_t (Q_{t+1} + A_{t+1})^2 \\ &= \rho_{1N} (Q_N(1) + A_N)^2 + \rho_{1B} (Q_B(1) + A_B)^2 \\ & \quad + \rho_{2N} (Q_N(2) + A_N)^2 + \rho_{2B} (Q_B(2) + A_B)^2 \end{aligned} \tag{2}$$

We have

$$\text{var}_t (Q_{t+1} + A_{t+1}) = E_t (Q_{t+1} + A_{t+1})^2 - (E_t (Q_{t+1} + A_{t+1}))^2 \tag{3}$$

Here

$$\begin{aligned} & (E_t (Q_{t+1} + A_{t+1}))^2 \\ &= (\rho_{1N})^2 (Q_N(1) + A_N)^2 + (\rho_{1B})^2 (Q_B(1) + A_B)^2 \\ & \quad + (\rho_{2N})^2 (Q_N(2) + A_N)^2 + (\rho_{2B})^2 (Q_B(2) + A_B)^2 \\ & \quad + 2\rho_{1N}\rho_{1B} (Q_N(1) + A_N) (Q_B(1) + A_B) \end{aligned} \tag{4}$$

$$\begin{aligned}
& +2\rho_{1N}\rho_{2N} (Q_N (1) + A_N) (Q_N (2) + A_N) \\
& +2\rho_{1N}\rho_{2B} (Q_N (1) + A_N) (Q_B (2) + A_B) \\
& +2\rho_{1B}\rho_{2N} (Q_B (1) + A_B) (Q_N (2) + A_N) \\
& +2\rho_{1B}\rho_{2B} (Q_B (1) + A_B) (Q_B (2) + A_B) \\
& +2\rho_{2N}\rho_{2B} (Q_N (2) + A_N) (Q_B (2) + A_B)
\end{aligned}$$

We then have

$$\begin{aligned}
& var_t (Q_{t+1} + A_{t+1}) \\
= & \rho_{1N} (1 - \rho_{1N}) (Q_N (1) + A_N)^2 + \rho_{1B} (1 - \rho_{1B}) (Q_B (1) + A_B)^2 \\
& + \rho_{2N} (1 - \rho_{2N}) (Q_N (2) + A_N)^2 + \rho_{2B} (1 - \rho_{2B}) (Q_B (2) + A_B)^2 \quad (5) \\
& - 2\rho_{1N}\rho_{1B} (Q_N (1) + A_N) (Q_B (1) + A_B) \\
& - 2\rho_{1N}\rho_{2N} (Q_N (1) + A_N) (Q_N (2) + A_N) \\
& - 2\rho_{1N}\rho_{2B} (Q_N (1) + A_N) (Q_B (2) + A_B) \\
& - 2\rho_{1B}\rho_{2N} (Q_B (1) + A_B) (Q_N (2) + A_N) \\
& - 2\rho_{1B}\rho_{2B} (Q_B (1) + A_B) (Q_B (2) + A_B) \\
& - 2\rho_{2N}\rho_{2B} (Q_N (2) + A_N) (Q_B (2) + A_B)
\end{aligned}$$

We can slightly simplify this as follows (as  $\rho_{1N} + \rho_{1B} + \rho_{2N} + \rho_{2B} = 1$ ):

$$\begin{aligned}
& var_t (Q_{t+1} + A_{t+1}) \\
= & \rho_{1N}\rho_{1B} [(Q_N (1) + A_N) - (Q_B (1) + A_B)]^2 \\
& + \rho_{1N}\rho_{2N} [(Q_N (1) + A_N) - (Q_N (2) + A_N)]^2 \quad (6) \\
& + \rho_{1N}\rho_{2B} [(Q_N (1) + A_N) - (Q_B (2) + A_B)]^2 \\
& + \rho_{1B}\rho_{2N} [(Q_B (1) + A_B) - (Q_N (2) + A_N)]^2 \\
& + \rho_{1B}\rho_{2B} [(Q_B (1) + A_B) - (Q_B (2) + A_B)]^2 \\
& + \rho_{2N}\rho_{2B} [(Q_N (2) + A_N) - (Q_B (2) + A_B)]^2
\end{aligned}$$

The market clearing condition then becomes

$$\begin{aligned}
& \rho_{1N} (Q_N (1) + A_N) + \rho_{1B} (Q_B (1) + A_B) \\
& + \rho_{2N} (Q_N (2) + A_N) + \rho_{2B} (Q_B (2) + A_B) - RQ \\
= & \lambda\rho_{1N}\rho_{1B} [(Q_N (1) + A_N) - (Q_B (1) + A_B)]^2
\end{aligned}$$

$$\begin{aligned}
& +\lambda\rho_{1N}\rho_{2N} [(Q_N(1) + A_N) - (Q_N(2) + A_N)]^2 \\
& +\lambda\rho_{1N}\rho_{2B} [(Q_N(1) + A_N) - (Q_B(2) + A_B)]^2 \\
& +\lambda\rho_{1B}\rho_{2N} [(Q_B(1) + A_B) - (Q_N(2) + A_N)]^2 \\
& +\lambda\rho_{1B}\rho_{2B} [(Q_B(1) + A_B) - (Q_B(2) + A_B)]^2 \\
& +\lambda\rho_{2N}\rho_{2B} [(Q_N(2) + A_N) - (Q_B(2) + A_B)]^2
\end{aligned} \tag{7}$$

where  $\lambda = \gamma K/W$ .

## 1.2 State N,1

Assume that we are currently in state  $(N, 1)$ . This implies:

$$\begin{aligned}
\rho_{1N} &= p_1 p_N \\
\rho_{1B} &= p_1 (1 - p_N) \\
\rho_{2N} &= (1 - p_1) p_N \\
\rho_{2B} &= (1 - p_1) (1 - p_N)
\end{aligned}$$

(7) becomes:

$$\begin{aligned}
& p_1 p_N (Q_N(1) + A_N) + p_1 (1 - p_N) (Q_B(1) + A_B) \\
& + (1 - p_1) p_N (Q_N(2) + A_N) + (1 - p_1) (1 - p_N) (Q_B(2) + A_B) - RQ_N(1) \\
= & \lambda (p_1)^2 p_N (1 - p_N) [(Q_N(1) + A_N) - (Q_B(1) + A_B)]^2 \\
& + \lambda p_1 (1 - p_1) (p_N)^2 [(Q_N(1) + A_N) - (Q_N(2) + A_N)]^2 \\
& + \lambda p_1 (1 - p_1) p_N (1 - p_N) [(Q_N(1) + A_N) - (Q_B(2) + A_B)]^2 \\
& + \lambda p_1 (1 - p_1) p_N (1 - p_N) [(Q_B(1) + A_B) - (Q_N(2) + A_N)]^2 \\
& + \lambda p_1 (1 - p_1) (1 - p_N)^2 [(Q_B(1) + A_B) - (Q_B(2) + A_B)]^2 \\
& + \lambda (1 - p_1)^2 p_N (1 - p_N) [(Q_N(2) + A_N) - (Q_B(2) + A_B)]^2
\end{aligned} \tag{8}$$

## 1.3 State B,1

Assume that we are currently in state  $(B, 1)$ . Then:

$$\begin{aligned}
\rho_{1N} &= p_1 (1 - p_B) \\
\rho_{1B} &= p_1 p_B
\end{aligned}$$

$$\begin{aligned}\rho_{2N} &= (1 - p_1)(1 - p_B) \\ \rho_{2B} &= (1 - p_1)p_B\end{aligned}$$

(7) becomes:

$$\begin{aligned}& p_1(1 - p_B)(Q_N(1) + A_N) + p_1p_B(Q_B(1) + A_B) \\ & + (1 - p_1)(1 - p_B)(Q_N(2) + A_N) + (1 - p_1)p_B(Q_B(2) + A_B) - RQ_B(1) \\ = & \lambda(p_1)^2 p_B(1 - p_B)[(Q_N(1) + A_N) - (Q_B(1) + A_B)]^2 \\ & + \lambda p_1(1 - p_1)(1 - p_B)^2 [(Q_N(1) + A_N) - (Q_N(2) + A_N)]^2 \\ & + \lambda p_1(1 - p_1)p_B(1 - p_B)[(Q_N(1) + A_N) - (Q_B(2) + A_B)]^2 \\ & + \lambda p_1(1 - p_1)p_B(1 - p_B)[(Q_B(1) + A_B) - (Q_N(2) + A_N)]^2 \\ & + \lambda p_1(1 - p_1)(p_B)^2 [(Q_B(1) + A_B) - (Q_B(2) + A_B)]^2 \\ & + \lambda(1 - p_1)^2 p_B(1 - p_B)[(Q_N(2) + A_N) - (Q_B(2) + A_B)]^2\end{aligned}\tag{9}$$

## 1.4 State N,2

Assume that we are currently in state  $(N, 2)$ . Then:

$$\begin{aligned}\rho_{1N} &= (1 - p_2)p_N \\ \rho_{1B} &= (1 - p_2)(1 - p_N) \\ \rho_{2N} &= p_2p_N \\ \rho_{2B} &= p_2(1 - p_N)\end{aligned}$$

(7) becomes:

$$\begin{aligned}& (1 - p_2)p_N(Q_N(1) + A_N) + (1 - p_2)(1 - p_N)(Q_B(1) + A_B) \\ & + p_2p_N(Q_N(2) + A_N) + p_2(1 - p_N)(Q_B(2) + A_B) - RQ_N(2) \\ = & \lambda(1 - p_2)^2 p_N(1 - p_N)[(Q_N(1) + A_N) - (Q_B(1) + A_B)]^2 \\ & + \lambda p_2(1 - p_2)(p_N)^2 [(Q_N(1) + A_N) - (Q_N(2) + A_N)]^2 \\ & + \lambda p_2(1 - p_2)p_N(1 - p_N)[(Q_N(1) + A_N) - (Q_B(2) + A_B)]^2 \\ & + \lambda p_2(1 - p_2)p_N(1 - p_N)[(Q_B(1) + A_B) - (Q_N(2) + A_N)]^2 \\ & + \lambda p_2(1 - p_2)(1 - p_N)^2 [(Q_B(1) + A_B) - (Q_B(2) + A_B)]^2 \\ & + \lambda(p_2)^2 p_N(1 - p_N)[(Q_N(2) + A_N) - (Q_B(2) + A_B)]^2\end{aligned}\tag{10}$$

## 1.5 State B,2

Assume that we are currently in state  $(B, 2)$ . Then:

$$\begin{aligned}\rho_{1N} &= (1 - p_2)(1 - p_B) \\ \rho_{1B} &= (1 - p_2)p_B \\ \rho_{2N} &= p_2(1 - p_B) \\ \rho_{2B} &= p_2p_B\end{aligned}$$

(7) becomes:

$$\begin{aligned}& (1 - p_2)(1 - p_B)(Q_N(1) + A_N) + (1 - p_2)p_B(Q_B(1) + A_B) \\ & + p_2(1 - p_B)(Q_N(2) + A_N) + p_2p_B(Q_B(2) + A_B) - RQ_B(2) \\ = & \lambda(1 - p_2)^2 p_B(1 - p_B)[(Q_N(1) + A_N) - (Q_B(1) + A_B)]^2 \\ & + \lambda p_2(1 - p_2)(1 - p_B)^2 [(Q_N(1) + A_N) - (Q_N(2) + A_N)]^2 \\ & + \lambda p_2(1 - p_2)p_B(1 - p_B)[(Q_N(1) + A_N) - (Q_B(2) + A_B)]^2 \\ & + \lambda p_2(1 - p_2)p_B(1 - p_B)[(Q_B(1) + A_B) - (Q_N(2) + A_N)]^2 \\ & + \lambda p_2(1 - p_2)(p_B)^2 [(Q_B(1) + A_B) - (Q_B(2) + A_B)]^2 \\ & + \lambda(p_2)^2 p_B(1 - p_B)[(Q_N(2) + A_N) - (Q_B(2) + A_B)]^2\end{aligned}\tag{11}$$

## 1.6 Rewriting the system

Define  $A_D = A_N - A_B$ ,  $Q_D(1) = Q_N(1) - Q_B(1)$  and  $Q_D(2) = Q_N(2) - Q_B(2)$ , and  $p_D = p_B(1 - p_B) - p_N(1 - p_N)$ . We rewrite the system in terms of the difference in asset prices between the normal and bad states, depending on the sunspot switching,  $Q_D(1)$  and  $Q_D(2)$ , the asset price difference in the bad state between the two sunspots,  $Q_B(1) - Q_B(2)$ . Some useful expressions are:

$$\begin{aligned}[(Q_N(1) + A_N) - (Q_B(1) + A_B)]^2 &= [Q_D(1) + A_D]^2 \\ [(Q_N(1) + A_N) - (Q_N(2) + A_N)]^2 &= [Q_D(1) - Q_D(2)]^2 + [Q_B(1) - Q_B(2)]^2 \\ &\quad + 2[Q_D(1) - Q_D(2)][Q_B(1) - Q_B(2)] \\ [(Q_N(1) + A_N) - (Q_B(2) + A_B)]^2 &= [Q_D(1) + A_D]^2 + [Q_B(1) - Q_B(2)]^2 \\ &\quad + 2[Q_D(1) + A_D][Q_B(1) - Q_B(2)] \\ [(Q_B(1) + A_B) - (Q_N(2) + A_N)]^2 &= [Q_B(1) - Q_B(2)]^2 + [Q_D(2) + A_D]^2\end{aligned}$$

$$\begin{aligned}
& -2 [Q_B (1) - Q_B (2)] [Q_D (2) + A_D] \\
[(Q_B (1) + A_B) - (Q_B (2) + A_B)]^2 &= [Q_B (1) - Q_B (2)]^2 \\
[(Q_N (2) + A_N) - (Q_B (2) + A_B)]^2 &= [Q_D (2) + A_D]^2
\end{aligned}$$

(8) is then written as:

$$\begin{aligned}
& p_1 p_N (Q_N (1) + A_N) + p_1 (1 - p_N) (Q_B (1) + A_B) \\
& + (1 - p_1) p_N (Q_N (2) + A_N) + (1 - p_1) (1 - p_N) (Q_B (2) + A_B) - RQ_N (1) \\
= & \lambda p_1 p_N (1 - p_N) [Q_D (1) + A_D]^2 + \lambda (1 - p_1) p_N (1 - p_N) [Q_D (2) + A_D]^2 \\
& + \lambda p_1 (1 - p_1) [p_N [Q_D (1) - Q_D (2)] + [Q_B (1) - Q_B (2)]]^2 \tag{12}
\end{aligned}$$

(9) is then written as:

$$\begin{aligned}
& p_1 (1 - p_B) (Q_N (1) + A_N) + p_1 p_B (Q_B (1) + A_B) \\
& + (1 - p_1) (1 - p_B) (Q_N (2) + A_N) + (1 - p_1) p_B (Q_B (2) + A_B) - RQ_B (1) \\
= & \lambda p_1 p_B (1 - p_B) [Q_D (1) + A_D]^2 + \lambda (1 - p_1) p_B (1 - p_B) [Q_D (2) + A_D]^2 \\
& + \lambda p_1 (1 - p_1) [(1 - p_B) [Q_D (1) - Q_D (2)] + [Q_B (1) - Q_B (2)]]^2 \tag{13}
\end{aligned}$$

(10) is then written as:

$$\begin{aligned}
& (1 - p_2) p_N (Q_N (1) + A_N) + (1 - p_2) (1 - p_N) (Q_B (1) + A_B) \\
& + p_2 p_N (Q_N (2) + A_N) + p_2 (1 - p_N) (Q_B (2) + A_B) - RQ_N (2) \\
= & \lambda (1 - p_2) p_N (1 - p_N) [Q_D (1) + A_D]^2 + \lambda p_2 p_N (1 - p_N) [Q_D (2) + A_D]^2 \\
& + \lambda p_2 (1 - p_2) [p_N [Q_D (1) - Q_D (2)] + [Q_B (1) - Q_B (2)]]^2 \tag{14}
\end{aligned}$$

(11) is then written as:

$$\begin{aligned}
& (1 - p_2) (1 - p_B) (Q_N (1) + A_N) + (1 - p_2) p_B (Q_B (1) + A_B) \\
& + p_2 (1 - p_B) (Q_N (2) + A_N) + p_2 p_B (Q_B (2) + A_B) - RQ_B (2) \\
= & \lambda (1 - p_2) p_B (1 - p_B) [Q_D (1) + A_D]^2 + \lambda p_2 p_B (1 - p_B) [Q_D (2) + A_D]^2 \\
& + \lambda p_2 (1 - p_2) [(1 - p_B) [Q_D (1) - Q_D (2)] + [Q_B (1) - Q_B (2)]]^2 \tag{15}
\end{aligned}$$

## 1.7 The general 3-equations system

Take the difference between (12) and (13):

$$p_1 [1 - p_N - p_B] [Q_D (1) + A_D]$$



$$\begin{aligned}
& + (1 - p_1) [1 - p_N - p_B] [Q_D(2) + A_D] + RQ_D(1) \tag{16} \\
= & \lambda p_1 p_D [Q_D(1) + A_D]^2 + \lambda (1 - p_1) p_D [Q_D(2) + A_D]^2 \\
& + \lambda p_1 (1 - p_1) \left[ \begin{array}{c} [(1 - p_B)^2 - (p_N)^2] [Q_D(1) - Q_D(2)]^2 \\ + 2 [1 - p_N - p_B] [Q_D(1) - Q_D(2)] [Q_B(1) - Q_B(2)] \end{array} \right]
\end{aligned}$$

Take the difference between (14) and (15):

$$\begin{aligned}
& (1 - p_2) [1 - p_N - p_B] [Q_D(1) + A_D] \\
& + p_2 [1 - p_N - p_B] [Q_D(2) + A_D] + RQ_D(2) \tag{17} \\
= & \lambda (1 - p_2) p_D [Q_D(1) + A_D]^2 + \lambda p_2 p_D [Q_D(2) + A_D]^2 \\
& + \lambda p_2 (1 - p_2) \left[ \begin{array}{c} [(1 - p_B)^2 - (p_N)^2] [Q_D(1) - Q_D(2)]^2 \\ + 2 [1 - p_N - p_B] [Q_D(1) - Q_D(2)] [Q_B(1) - Q_B(2)] \end{array} \right]
\end{aligned}$$

Take the difference between (13) and (15):

$$\begin{aligned}
& [1 - p_1 - p_2] (1 - p_B) [Q_D(1) - Q_D(2)] + [1 + R - p_1 - p_2] [Q_B(1) - Q_B(2)] \\
= & \lambda [1 - p_1 - p_2] p_B (1 - p_B) \left( [Q_D(1) + A_D]^2 - [Q_D(2) + A_D]^2 \right) \tag{18} \\
& - \lambda [p_1 (1 - p_1) - p_2 (1 - p_2)] [(1 - p_B) [Q_D(1) - Q_D(2)] + [Q_B(1) - Q_B(2)]]^2
\end{aligned}$$

We thus have a non-linear system in 3 equations (16) - (18) in 3 unknowns:  $Q_D(1)$ ,  $Q_D(2)$ ,  $Q_B(1) - Q_B(2)$ . There are multiple solutions to this system.

### 1.7.1 The case of identical asset price difference: $Q_D(1) = Q_D(2)$

It appears that the only case that we can solve analytically is the one where  $Q_D(1) = Q_D(2)$ . We define:  $x = Q_D(1) + A_D = Q_D(2) + A_D$ , and  $\kappa = 1 + R - p_N - p_B$ . (16) and (17) then becomes the same equation:

$$\kappa x - RA_D = \lambda p_D x^2 \tag{19}$$

This has two solutions:

$$x = \frac{\kappa \pm (\kappa^2 - 4R\lambda p_D A_D)^{0.5}}{2\lambda p_D} \tag{20}$$

These are exactly the same values for  $Q_D + A_D$  as in the case where states 1 and 2 are the same (see the paper).

Given these solutions for  $x$  we can solve for  $Q_B(1) - Q_B(2)$  from (18). We get

$$\tilde{\kappa}(Q_B(1) - Q_B(2)) = \lambda\tilde{p}_D(Q_B(1) - Q_B(2))^2 \quad (21)$$

where  $\tilde{p}_D = p_2(1 - p_2) - p_1(1 - p_1)$  and  $\tilde{\kappa} = 1 + R - p_1 - p_2$ . This has two solutions. The first one is  $Q_B(1) = Q_B(2)$ . Together with  $Q_D(1) = Q_D(2)$  this means that in this solution the asset price does not depend on whether we are in state 1 or 2. It only depends on whether we are in state  $N$  or  $B$ . In this case the solution for  $Q_B$  (same in state 1 and 2) is the one listed in Proposition 5. There are therefore two equilibria in this case: the fundamental and sunspot-like equilibria of proposition 5, regardless of state 1 or 2.

The other solution is

$$Q_B(1) - Q_B(2) = \frac{\tilde{\kappa}}{\lambda\tilde{p}_D} \quad (22)$$

This is like the sunspot equilibrium with a Markov process, with states  $N$  and  $B$  now replaced by states 1 and 2. In this case there are two equilibria. In both equilibria the asset price drops by an amount  $Q_D$  when going from state  $N$  to state  $B$  and by an amount  $Q_B(1) - Q_B(2)$  when going from state 1 to state 2. For  $Q_D$  there are two possible values, so there are two equilibria. Consider the case where  $A_D = 0$ . Then in the first equilibrium the asset price is the same in states  $N$  and  $B$ . States 1 and 2 then operate like a sunspot with the asset price dropping in state 2. This is like the reverse of the previous equilibrium where the asset price is the same in states 1 and 2 while  $N$  and  $B$  operate like a sunspot. In the other equilibrium there is a double sunspot with the asset price dropping both when we switch from  $N$  to  $B$  and when we switch from 1 to 2.

To summarize, we have 4 equilibria: the fundamental equilibrium; an equilibrium where states (1,2) operate like a sunspot and the price is the same in state  $N$  as in state  $B$ ; an equilibrium where states ( $N, B$ ) operate like a sunspot and the price is the same in state 1 as in state 2; and a double sunspot equilibrium. Notice that we implicitly assumed that  $0.5 < p_2 < p_1 < 1$ . If we assume  $p_1 = p_2$  then the last two equilibria no longer exist. States (1,2) cannot operate like a sunspot.

Two more points on these equilibria. When the last two equilibria exist we can compute the level of the asset prices using for example (12). Write  $Q_N(1) = x + h + Q_B(2)$ ,  $Q_N(2) = x + Q_B(2)$  and  $Q_B(1) = h + Q_B(2)$ , where  $x$  is one of the solutions for  $Q_D(1) = Q_D(2)$  and  $h$  is either of the two solutions for  $Q_B(1) - Q_B(2)$ .

Substituting this into (12) we get

$$Q_B(2) = \frac{1}{R-1} (p_N A_N + (1-p_N)A_B + (p_1 - R)x + (p_N - R)h) - \frac{\lambda}{R-1} (p_N(1-p_N)(x+A_D)^2 + p_1(1-p_1)h^2) \quad (23)$$

The other point to make is that these 4 solutions so far are not really switching solutions that we have in mind. We have in mind a solution where in state 2 (a bad state) the value of  $Q_D$  is much higher than in state 1 as it is like switching to the sunspot or sunspot-like equilibrium. In all the equilibria above  $Q_D$  is the same in state 1 and 2.

### 1.7.2 The case of different asset price difference: $Q_D(1) \neq Q_D(2)$

We now turn to equilibria where  $Q_D$  differs across state 1 and 2. In this case, we have to compute the solution numerically, following the algorithm derived below.

We now write  $x = Q_D(1) + A_D$ ,  $y = Q_D(2) + A_D$  and  $z = (x-y)(Q_B(1) - Q_B(2))$ . Note that since we now focus on solutions where  $x \neq y$ ,  $Q_B(1) - Q_B(2)$  follows from any solution for  $x$ ,  $y$  and  $z$ . Using this notation, we write (16) as:

$$\begin{aligned} & (p_1 [1 - p_N - p_B] + R)x + (1 - p_1) [1 - p_N - p_B]y - RA_D(1) \quad (24) \\ = & \lambda p_1 p_D x^2 + \lambda (1 - p_1) p_D y^2 + \lambda p_1 (1 - p_1) [(1 - p_B)^2 - (p_N)^2] (x - y)^2 \\ & + 2\lambda p_1 (1 - p_1) [1 - p_N - p_B] z \end{aligned}$$

This expression is of the form:

$$z = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad (25)$$

where the coefficients  $\alpha$  depend on  $y$ :

$$\alpha_1 = \frac{1}{2\lambda p_1} y - \frac{p_D + p_1 [(1 - p_B)^2 - (p_N)^2]}{2p_1 [1 - p_N - p_B]} y^2 - \frac{RA_D(1)}{2\lambda p_1 (1 - p_1) [1 - p_N - p_B]} \quad (26)$$

$$\alpha_2 = \frac{p_1 [1 - p_N - p_B] + R}{2\lambda p_1 (1 - p_1) [1 - p_N - p_B]} + \frac{(1 - p_B)^2 - (p_N)^2}{[1 - p_N - p_B]} y \quad (27)$$

$$\alpha_3 = -\frac{p_D + (1 - p_1) [(1 - p_B)^2 - (p_N)^2]}{2(1 - p_1) [1 - p_N - p_B]} \quad (28)$$

Next take (17), which we can write as:

$$0 = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 z \quad (29)$$

where the coefficients  $\beta$  depend on  $y$ :

$$\beta_1 = (p_2 [1 - p_N - p_B] + R) y - \lambda p_2 [p_D + (1 - p_2) [(1 - p_B)^2 - (p_N)^2]] y^2 - R A_D$$

$$\beta_2 = (1 - p_2) [1 - p_N - p_B] + 2\lambda p_2 (1 - p_2) [(1 - p_B)^2 - (p_N)^2] y$$

$$\beta_3 = -\lambda (1 - p_2) [p_D + p_2 [(1 - p_B)^2 - (p_N)^2]]$$

$$\beta_4 = -2\lambda p_2 (1 - p_2) [1 - p_N - p_B]$$

Finally consider (18). After multiplying it with  $(x - y)^2$ , it can be written as:

$$0 = \delta_1 (x - y)^3 + \delta_2 z (x - y) + \delta_3 (x + y) (x - y)^3 + \delta_4 [(1 - p_B) (x - y)^2 + z]^2 \quad (30)$$

where the coefficients  $\delta$  are:

$$\delta_1 = [1 - p_1 - p_2] (1 - p_B) \quad (31)$$

$$\delta_2 = 1 + R - p_1 - p_2 \quad (32)$$

$$\delta_3 = -\lambda [1 - p_1 - p_2] p_B (1 - p_B) \quad (33)$$

$$\delta_4 = \lambda [p_1 (1 - p_1) - p_2 (1 - p_2)] \quad (34)$$

(30) can be rewritten further as:

$$0 = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 x^4 + \gamma_5 z + \gamma_6 z^2 + \gamma_7 z x + \gamma_8 z x^2 \quad (35)$$

where:

$$\gamma_0 = -[\delta_3 - \delta_4 (1 - p_B)^2] y^4 - \delta_1 y^3 \quad (36)$$

$$\gamma_1 = 3\delta_1 y^2 + 2[\delta_3 - 2\delta_4 (1 - p_B)^2] y^3 \quad (37)$$

$$\gamma_2 = 6\delta_4 (1 - p_B)^2 y^2 - 3\delta_1 y \quad (38)$$

$$\gamma_3 = \delta_1 - 2[\delta_3 + 2\delta_4 (1 - p_B)^2] y \quad (39)$$

$$\gamma_4 = \delta_3 + \delta_4 (1 - p_B)^2 \quad (40)$$

$$\gamma_5 = 2\delta_4 (1 - p_B) y^2 - \delta_2 y \quad (41)$$

$$\gamma_6 = \delta_4 \quad (42)$$

$$\gamma_7 = \delta_2 - 4\delta_4 (1 - p_B) y \quad (43)$$

$$\gamma_8 = 2\delta_4 (1 - p_B) \quad (44)$$

(25) gives us  $z$  as a quadratic function of  $x$  (conditional on  $y$ ). Using it into (29) gives a quadratic function if  $x$ . Using it into (30) gives a fourth polynomial in  $x$ .

Specifically, combining (25) and (29) gives:

$$0 = (\beta_1 + \beta_4\alpha_1) + (\beta_2 + \beta_4\alpha_2)x + (\beta_3 + \beta_4\alpha_3)x^2 \quad (45)$$

Combining (25) and (35) gives:

$$\begin{aligned} 0 = & \left[ \gamma_0 + \gamma_5\alpha_1 + \gamma_6\alpha_1^2 \right] \\ & + \left[ \gamma_1 + \gamma_5\alpha_2 + \gamma_7\alpha_1 + 2\gamma_6\alpha_1\alpha_2 \right] x \\ & + \left[ \gamma_2 + \gamma_5\alpha_3 + \gamma_7\alpha_2 + \gamma_8\alpha_1 + \gamma_6 \left[ 2\alpha_1\alpha_3 + \alpha_2^2 \right] \right] x^2 \\ & + \left[ \gamma_3 + \gamma_7\alpha_3 + \gamma_8\alpha_2 + 2\gamma_6\alpha_2\alpha_3 \right] x^3 \\ & + \left[ \gamma_4 + \gamma_8\alpha_3 + \gamma_6\alpha_3^2 \right] x^4 \end{aligned} \quad (46)$$

Write these polynomials as

$$h_1x^2 + h_2x + h_3 = 0 \quad (47)$$

and

$$g_1x^4 + g_2x^3 + g_3x^2 + g_4x + g_5 = 0 \quad (48)$$

From the quadratic polynomial we have the solutions

$$x = \frac{-h_2 \pm (h_2^2 - 4h_1h_3)^{0.5}}{2h_1} \quad (49)$$

For each of these two solutions for  $x$ , consider the following function of  $y$ . For a given  $y$  solve for  $x$  (one of the two solutions above). Then substitute the result into the fourth-order polynomial. Call the result  $f(y)$ . Plot this. Look at the case where this function is zero. This is a solution. In following this procedure we need to make sure to rule out solutions where  $x = y$  as we have already considered those above. So when one of the solutions to the quadratic polynomial is such that  $x = y$ , do not consider that.

## 1.8 The case of identical persistence in states 1 and 2: $p_1 = p_2 = p$

### 1.8.1 Main results

We assume that  $p_1 = p_2 = p > 0.5$ . (16) is:

$$\begin{aligned}
& (p[1 - p_N - p_B] + R)x + (1 - p)[1 - p_N - p_B]y - RA_D \quad (50) \\
& = \lambda p_D p x^2 + \lambda p_D (1 - p) y^2 \\
& \quad + \lambda p (1 - p) \left[ \begin{array}{c} [(1 - p_B)^2 - (p_N)^2] (x - y)^2 \\ + 2[1 - p_N - p_B] (x - y) [Q_B(1) - Q_B(2)] \end{array} \right]
\end{aligned}$$

where  $x = Q_D(1) + A_D$  and  $y = Q_D(2) + A_D$ . (17) is:

$$\begin{aligned}
& (1 - p)[1 - p_N - p_B]x + (p[1 - p_N - p_B] + R)y - RA_D \quad (51) \\
& = \lambda p_D (1 - p) x^2 + \lambda p_D p y^2 \\
& \quad + \lambda p (1 - p) \left[ \begin{array}{c} [(1 - p_B)^2 - (p_N)^2] (x - y)^2 \\ + 2[1 - p_N - p_B] (x - y) [Q_B(1) - Q_B(2)] \end{array} \right]
\end{aligned}$$

And (18) is:

$$\begin{aligned}
& (2p - 1)(1 - p_B)(x - y) - [R - (2p - 1)][Q_B(1) - Q_B(2)] \\
& = \lambda(2p - 1)p_B(1 - p_B)(x + y)(x - y) \quad (52)
\end{aligned}$$

Take the difference between (50) and (51):

$$\begin{aligned}
& (R + (2p - 1)[1 - p_N - p_B])(x - y) \quad (53) \\
& = \lambda(2p - 1)p_D(x + y)(x - y)
\end{aligned}$$

One solution of (53) is  $x = y$ . (50) or (51) then imply:

$$0 = \lambda p_D x^2 - \kappa x + RA_D \quad (54)$$

$$x = \frac{\kappa \pm [\kappa^2 - 4R\lambda p_D A_D]^{0.5}}{2\lambda p_D} \quad (55)$$

(52) implies  $Q_B(1) - Q_B(2) = 0$ . In this case the asset price is the same in state 1 as in state 2. The asset price only depends on whether we are in  $N$  or  $B$ . This leads to the well known fundamental and sunspot(-like) equilibria that we have already computed in Proposition 6.

In the other solution,  $x \neq y$  (53) implies:

$$x + y = \frac{R + (2p - 1) [1 - p_N - p_B]}{\lambda (2p - 1) p_D} \quad (56)$$

Using that  $x = 0.5(x + y) + 0.5(x - y)$  and  $y = 0.5(x + y) - 0.5(x - y)$ , we write (52) as:

$$Q_B(1) - Q_B(2) = -\delta(x - y) \quad (57)$$

where

$$\delta = (1 - p_B) \frac{(2p - 1) p_N (1 - p_N - p_B) + p_B R}{p_D (1 + R - 2p)} \quad (58)$$

Next take the sum of (50) and (51):

$$\begin{aligned} \kappa(x + y) - 2RA_D &= \frac{1}{2} \lambda p_D (x - y)^2 + \frac{1}{2} \lambda p_D (x + y)^2 \\ &+ 2\lambda p(1 - p)(1 - p_N - p_B)(1 - p_B + p_N - 2\delta)(x - y)^2 \end{aligned} \quad (59)$$

The solution is:

$$x - y = \pm \left( \frac{\kappa(x + y) - 2RA_D - \frac{1}{2} \lambda p_D (x + y)^2}{\frac{1}{2} \lambda p_D + 2\lambda p(1 - p)(1 - p_N - p_B)(1 - p_B + p_N - 2\delta)} \right)^{0.5} \quad (60)$$

It is instructional to consider the case where  $p \rightarrow 1$ . In that case we have

$$x + y = \frac{\kappa}{\lambda p_D} \quad (61)$$

which gives:

$$x - y = \pm \frac{(\kappa^2 - 4\lambda p_D RA_D)^{0.5}}{\lambda p_D} \quad (62)$$

We thus have two equilibria, but they are really the same as  $y$  in one corresponds to  $y$  in the other. Specifically,  $x$  (or  $y$ ) is identical to (55):

$$x \text{ (or } y) = \frac{\kappa \pm [\kappa^2 - 4R\lambda p_D A_D]^{0.5}}{2\lambda p_D}$$

This is exactly the difference of  $Q_D$  in the fundamental and sunspot-like equilibria. In this case the switching equilibrium indeed becomes a switch between the fundamental and sunspot-like equilibria. It is easy to check from proposition 6 that  $Q_B(1) - Q_B(2)$  in this case is also exactly the same as the difference between the fundamental and sunspot-like equilibria.

So we have four possible outcome. In the first  $x$  and  $y$  are equal to the expression (55) with the negative sign. In the second they are equal to the expression (55) with the positive sign. In these two equilibria, state 1 and 2 are the same. In the third equilibrium,  $x$  is equal to the expression (55) with the negative sign and  $y$  is equal to the expression (55) with the positive sign. The opposite holds in the fourth equilibrium. In these last two equilibrium, the state 1 and 2 differ as each correspond to one of the two solutions in (55).

### 1.8.2 Restrictions: bracket in (60)

We first make sure that the expression in brackets in (60) is positive, so that we can indeed take the square root. Start with the denominator. We can show that  $\delta > 1$ . In addition it is an increasing function of  $p$ :

$$\frac{\partial \delta}{\partial p} = \frac{1}{2} \frac{\partial \delta}{\partial (2p-1)} = \frac{(1-p_B)R(1-p_N)(p_B+p_N)}{2p_D(2p-1-R)^2} > 0 \quad (63)$$

Therefore  $\delta$  takes its smallest value at  $p = 0.5$  (we assume  $p > 0.5$ ). It is then sufficient to show that  $\delta > 1$  at  $p = 0.5$ . This is the case when  $(1-p_B)p_B/p_D > 1$ , which is clearly the case as  $p_D < p_B(1-p_B)$ . Therefore  $\delta > 1$  and thus  $1-p_B+p_N-2\delta < -p_B-(1-p_N) < 0$ , so that the denominator of (60) is positive as  $1-p_N-p_B < 0$  because both  $p_N$  and  $p_B$  are above 0.5.

The numerator of the term in brackets in (60) needs to be positive as well, which implies:

$$A_D < \frac{1}{2R} \left[ \kappa(x+y) - \frac{1}{2} \lambda p_D (x+y)^2 \right] \quad (64)$$

Using (56) we rewrite this condition as:

$$A_D < \bar{A}_D = \frac{R + (2p-1)[1-p_N-p_B]}{4R\lambda(2p-1)p_D} \left[ \frac{4p-3}{2p-1} R + [1-p_N-p_B] \right] \quad (65)$$

As  $A_D$  cannot be negative, a necessary (but not sufficient) condition for this inequality to hold is that the right-hand side be positive, which implies:

$$p > \bar{p} = \frac{3R+1-p_N-p_B}{4R+2-2p_N-2p_B} \quad (66)$$

This means that we need at least  $p > 0.75$ . If  $R = 1.1$ ,  $p_N = 0.95$  and  $p_B = 0.7$ , for example, then we need  $p > 0.85$ . So clearly,  $p$  will need to be sufficiently close



to 1 for a switching equilibrium to exist. Note that if  $p = 1$  the right-hand side of (64) is maximized at  $\kappa^2 (4R\lambda p_D)^{-1}$  which is the threshold in proposition 6. At this point, we have derived the conditions that ensure that the expression in brackets in (60) is positive.

### 1.8.3 Restrictions: asset prices

We next consider the level of the asset prices. We show below that  $Q_B(2)$  is the lowest price in the switching equilibrium if we assume that state 2 is the bad state. Define  $u = Q_B(1) - Q_B(2)$ . This implies  $Q_B(1) + A_B = Q_B(2) + A_B + u$ , and  $Q_N(2) + A_N = Q_B(2) + A_B + y$ , and  $Q_N(1) + A_N = Q_B(2) + A_B + u + x$ . Substituting these into (12) gives:

$$Q_B(2) = \frac{A_B + \nu}{R - 1} \quad (67)$$

where:

$$\begin{aligned} \nu &= RA_D + (p - R)u + (pp_N - R)x + (1 - p)p_Ny \\ &\quad - \lambda p_N(1 - p_N)(px^2 + (1 - p)y^2) \\ &\quad - \lambda p(1 - p)(p_N(x - y) + u)^2 \\ &= RA_D + (p - R)(Q_B(1) - Q_B(2)) \\ &\quad + (p_N - R)\frac{1}{2}(x + y) + ((2p - 1)p_N - R)\frac{1}{2}(x - y) \\ &\quad - \lambda p_N(1 - p_N)\left(\frac{1}{4}(x + y)^2 + \frac{1}{4}(Q_D(1) - Q_D(2))^2\right. \\ &\quad \left. + \frac{2p-1}{2}(x + y)(Q_D(1) - Q_D(2))\right) \\ &\quad - \lambda p(1 - p)(p_N(Q_D(1) - Q_D(2)) + (Q_B(1) - Q_B(2)))^2 \end{aligned} \quad (69)$$

and above  $x - y$  is the negative root in (60).  $A_B > \bar{A}_B$  ensures that this is positive.

We now show that  $Q_B(2)$  is indeed the lowest asset price in the switching equilibrium. As state 2 corresponds to the negative root in (60) we have  $x - y < 0$  in state 2. Since  $\delta > 0$ , it then follows that  $Q_B(1) > Q_B(2)$ . As  $x - y < 0$  we know that  $Q_D(2) > Q_D(1)$ . If  $Q_D(1) > 0$  this implies  $Q_D(2) > 0$ , so that  $Q_N(2) > Q_B(2)$ . Also,  $Q_D(1) > 0$  implies  $Q_N(1) > Q_B(1) > Q_B(2)$ . Thus all we need to show to establish that  $Q_B(2)$  is the lowest asset price is that  $Q_D(1) > 0$ .

We start by rewriting  $Q_D(1) > 0$  as follows:

$$Q_D(1) > 0$$

$$\begin{aligned}
(Q_D(1) - Q_D(2)) + (Q_D(1) + Q_D(2)) &> 0 \\
(x - y) + (x + y) - 2A_D &> 0 \\
(x + y) - 2A_D &> y - x
\end{aligned} \tag{70}$$

We know that in state 2  $y - x > 0$ . (64) implies:

$$(x + y) - 2A_D > \frac{R - \kappa}{R}(x + y) + \frac{1}{2R}\lambda p_D(x + y)^2 > 0$$

The left-hand side of (70) is thus positive, and we can square both sides:

$$(x + y)^2 + 4A_D^2 - 4(x + y)A_D > (y - x)^2 \tag{71}$$

Notice that it is sufficient to show that

$$(x + y)^2 - 4(x + y)A_D > (y - x)^2 \tag{72}$$

We know from (60) that:

$$(y - x)^2 = \frac{\kappa(x + y) - 2RA_D - \frac{1}{2}\lambda p_D(x + y)^2}{\frac{1}{2}\lambda p_D + 2\lambda p(1 - p)(1 - p_N - p_B)(1 - p_B + p_N - 2\delta)} \tag{73}$$

$$< \frac{\kappa(x + y) - 2RA_D - \frac{1}{2}\lambda p_D(x + y)^2}{\frac{1}{2}\lambda p_D} \tag{74}$$

where the second inequality uses  $(1 - p_N - p_B)(1 - p_B + p_N - 2\delta) > 0$  as shown above. A sufficient condition for (72) to be satisfied is then:

$$\frac{\kappa(x + y) - 2RA_D - \frac{1}{2}\lambda p_D(x + y)^2}{\frac{1}{2}\lambda p_D} < (x + y)^2 - 4(x + y)A_D \tag{75}$$

which we rewrite as:

$$[\lambda p_D(x + y) - \kappa][x + y - 2A_D] + 2[p_N + p_B - 1]A_D > 0 \tag{76}$$

(76) holds if  $[\lambda p_D(x + y) - \kappa](x + y - 2A_D)$  is positive. The first term is clearly positive:

$$\lambda p_D(x + y) - \kappa = \frac{2(1 - p)}{2p - 1}R > 0 \tag{77}$$

Using (64) we write the second term as:

$$\begin{aligned}
x + y - 2A_D &> (x + y) \left[ \left(1 - \frac{\kappa}{R}\right) + \frac{\lambda p_D}{2R}(x + y) \right] \\
&= (x + y) \left[ \frac{p_N + p_B - 1}{R} + \frac{\lambda p_D}{2R}(x + y) \right] > 0
\end{aligned} \tag{78}$$

Which proves that  $Q_D(1) > 0$ , which in turn completes the proof that  $Q_B(2)$  is the lowest asset price. As an aside, is now also easy to show that  $Q_N(1) > Q_N(2)$ . To see this, use that  $Q_N(1) - Q_N(2) = Q_B(1) - Q_B(2) + (Q_D(1) - Q_D(2)) = (1 - \delta)(x - y)$ . Since  $x - y < 0$  and  $\delta > 1$ , this is positive.

To summarize, the asset price unambiguously drops when we go to state 2. It drops more when we are in state  $B$ , so when the fundamental is weak. Also, starting in state  $(1, N)$ , a deterioration of the fundamental leads to a drop in the asset price. As we will show numerically, this drop can be small relative to the additional drop when we hit the panic button (go to state 2). A panic leads to a larger drop in the asset price when the fundamental is weak (which follows from  $Q_D(2) > Q_D(1)$ , so that  $Q_B(2) - Q_B(1) < Q_N(2) - Q_N(1)$ ).

## 2 Note on Symmetry

Normalizing the unconditional mean of the sunspot  $S_t$  around zero, we can define a symmetric conditional distribution for the sunspot as follows. Index the values that  $S_t$  can take on as  $s_j$  with  $j \in J$ . The set  $J$  can potentially be infinite. Assume that when  $S_t$  can take on the value  $s_j$ , it can also take on the value  $-s_j$ , so that the set  $\{s_j\}$  with  $j \in J$  is the same as the same set  $\{-s_j\}$  with  $j \in J$ . We define conditional symmetry as

$$\text{prob}(S_{t+1} = s_i | S_t = s_j) = \text{prob}(S_{t+1} = -s_i | S_t = -s_j) \quad (79)$$

This is clearly satisfied for the AR process combined with a symmetrically distributed innovation as

$$\text{prob}(S_{t+1} = \rho\alpha_i + \epsilon_i | S_t = \alpha_i) = \text{prob}(S_{t+1} = -\rho\alpha_i - \epsilon_i | S_t = -\alpha_i) \quad (80)$$

because the probability of  $\epsilon_{t+1} = \epsilon_i$  is equal to the probability of  $\epsilon_{t+1} = -\epsilon_i$ . It is clearly not satisfied for the asymmetric 2-state Markov process, where we can think of  $S_N = 1$  and  $S_B = -1$  because

$$\text{prob}(S_{t+1} = S_N | S_t = S_N) = p_N \neq p_B = \text{prob}(S_{t+1} = -S_N | S_t = -S_N) \quad (81)$$

It would be nice to be able to show that if a sunspot equilibrium  $Q_t = f(S_t)$  exists, and the conditional distribution of the sunspot is symmetric, then the sunspot

solution for the price must be symmetric as well, so that  $f(S_t) = f(-S_t)$ . I have not been able to prove this. I can prove the weaker result that within the loop from the price to risk and back to the price again, symmetry is sustained. In other words, when the asset price is a symmetric function of the sunspot, so will risk and when risk is a symmetric function of the sunspot, so will the price. This is weaker because it does not rule out that at the same time asymmetric price solutions exist. And indeed, for the autoregressive process there may be many other sunspot solutions, perhaps some not symmetric.

I will first show that  $Risk_t$  is a symmetric function of  $S_t$  when  $Q_t = f(S_t)$  is a symmetric function of  $S_t$ , assuming a symmetric conditional distribution of  $S_{t+1}$ . We have

$$\begin{aligned}
Risk_t(S_t = s_i) &= var_t(Q_{t+1}|S_t = s_i) = \\
&\sum_{j \in J} prob(S_{t+1} = s_j|S_t = s_i)f^2(s_j) - \left( \sum_{j \in J} prob(S_{t+1} = s_j|S_t = s_i)f(s_j) \right)^2 = \\
&\sum_{j \in J} prob(S_{t+1} = -s_j|S_t = -s_i)f^2(-s_j) - \left( \sum_{j \in J} prob(S_{t+1} = -s_j|S_t = -s_i)f(-s_j) \right)^2 = \\
var_t(Q_{t+1}|S_t = -s_i) &= Risk_t(S_t = -s_i) \tag{82}
\end{aligned}$$

Next I show that when  $Risk_t$  is a symmetric function of  $S_t$ , it implies that  $Q_t$  is a symmetric function of  $S_t$ . We know that  $Q_t$  depends on  $Risk_t$  and expectations of  $Risk_{t+i}$ ,  $i > 0$ . So we need to show that the expectation of  $Risk_{t+i}$ ,  $i > 0$ , is symmetric in  $S_t$  when  $Risk_{t+i}$  is symmetric in  $S_{t+i}$ . Since  $E_t Risk_{t+i} = E_t E_{t+1} \dots E_{t+i-1} Risk_{t+i}$  it is sufficient to show that when a variable  $x_{t+1}$  is a symmetric function of  $S_{t+1}$ , then the expectation at time  $t$  (one period earlier) of that variable is symmetric in  $S_t$ . Backward induction then gets our result. This is need the case:

$$\begin{aligned}
E_t(x_{t+1}|S_t = s_i) &= \sum_{j \in J} prob(S_{t+1} = s_j|S_t = s_i)x(s_j) = \\
\sum_{j \in J} prob(S_{t+1} = -s_j|S_t = -s_i)x(-s_j) &= E_t(x_{t+1}|S_t = -s_i) \tag{83}
\end{aligned}$$

### 3 Equity Premium Puzzle

In the standard consumption asset pricing model with constant relative risk-aversion the equity premium is equal to

$$-cov((C_{t+1}/C_t)^{-\gamma}, R_{K,t+1}) \quad (84)$$

With  $g_{c,t+1}$  denoting consumption growth, this can be approximated as

$$\gamma cov(g_{c,t+1}, R_{K,t+1}) \quad (85)$$

Since the equity return is not highly correlated with consumption growth, this leads to a low equity premium.

Instead we use mean-variance preferences, but our equity premium formula is very similar to the one above. We cannot compute a consumption growth rate since agents consume only one period. But think of one plus consumption growth in our model as consumption at  $t + 1$  relative to what it would be if the portfolio return were 1. Then consumption growth is simply  $R_{t+1}^p$ , which is wealth growth. Note that in the standard model consumption is proportional to wealth, so consumption and wealth growth are the same anyway. Applying the formula above, the equity premium is

$$\gamma cov(R_{t+1}^p, R_{K,t+1}) = \gamma \alpha_t var(R_{K,t+1}) = \gamma \frac{\alpha_t var_t(A_{+1} + Q_{t+1})}{Q_t} \quad (86)$$

Using the market clearing condition  $\alpha_t W = K Q_t$ , the last equation is

$$\frac{\gamma K var_t(A_{+1} + Q_{t+1})}{W Q_t} \quad (87)$$

Using  $\gamma K/W = \lambda$  and equation (6) in the paper, this becomes

$$E_t \frac{A_{t+1} + Q_{t+1}}{Q_t} - R \quad (88)$$

which is indeed the equity premium in our paper. This goes to show that the equity premium formula in our paper is the standard one.

We do not really run into a Mehra Prescott puzzle though since in our model consumption only depends on the return on the risky equity. The correlation

between consumption and the equity return is then 1. Of course aggregate consumption and the equity return have a correlation far below 1, which is what causes the equity premium puzzle.

The puzzle does not arise in our paper. For example, in Figure 1 the average equity premium ranges from 2.1% for  $p_B=0.73$  to 9% for  $p_B=0.92$ . This is because consumption and the equity return are perfectly correlated. Note though that the equity premium is tiny in the normal state compared to the bad state. In the numbers above I computed the average equity premium, using that we spent a fraction  $(1 - p_B)/(2 - p_N - p_B)$  in the normal state. The equity premium ranges from about 1 to 2% in the normal state (dependent on  $p_B$ ) and from about 30% to about 100% in the bad state. For example, for  $p_B = 0.7$  the equity premium is 1% in the normal state, 38% in the bad state and 2.1% on average.