Technical Appendix for A Method for Solving DSGE Portfolio Choice Models with Dispersed Private Information

Cedric Tille
Geneva Graduate Institute of International and Development Studies
CEPR

Eric van Wincoop
University of Virginia
NBER

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1 Introduction

This Technical Appendix presents the detailed solution of the two country general equilibrium model. We proceed in three broad steps: we first lay out and expand the various equations, then compute the returns on assets and the various agents' expectations of these returns, and finally impose the equations to various orders in order to derive the solution of the control variables as a function of state variables to various orders.

The first step covers Sections 2 through 4. Section 2 presents the various equations with the variables in logarithmic form and combines some equations to derive a more compact system. Sections 3 and 4 expand the system around the zero-order allocation. The approximated system consists of six control variables (the future capital stocks, the current asset prices and the current portfolio shares, each in terms of worldwide average and cross-country difference) and four state variables (the current capital stocks and technology, also in terms of worldwide average and cross-country difference). The variables are linked through six non-linear equations (Tobin's Q relation that determines the future capital stock, asset market clearing conditions, and the Euler conditions for portfolio choice). Like the variables, for the equations we also take the world wide averages and cross country differences. Section 3 presents the quadratic approximations of the Tobin's Q and asset market clearing equations. Section 4 derives the cubic approximation of the portfolio Euler equations.

The second step of our analysis focuses on the excess return, which enters the portfolio Euler equations. This is covered in Sections 5 to 7. Section 5 derives a cubic approximation of this excess return as a function of current state variables and future innovations. Section 6 solves the information extraction problem where individual agents form their expectations of future innovations using both their noisy private information on future innovations, as well as the publicly observed asset prices. Section 7 uses these individual investors' expectations to compute the first, second and third-order components of the expected excess return, as well as the quadratic and cubic excess return.

The final stage of the analysis in section 8 draws on these building blocks to impose the equations to various orders and solve for the control variables to various orders. We first derive the solution for the zero-order cross-country difference in portfolio shares, along with the first-order solution for all other variables. This is done using the second-order component of the cross-country difference between the portfolio Euler equations, along with the first-order component of all other equations. This solution remains partial, as it is conditional on the signal to noise ratio in agents' signals. The next step repeats this to one higher order. Specifically, we get the solution for the first-order cross-country difference in portfolio shares and the second-order solution for all other variables. This is done using the third-order component of the cross-country difference between the portfolio Euler equations and the second-order component of all other equations. Again, the solution is conditional on the signal to noise ratio in agents' signals.

The final step solves for the signal to noise ratio by combining the third-order component of the worldwide average of the portfolio Euler equation, which drives global asset demand (first-order component of average portfolio share) and the first-order component of the average asset market clearing condition, which gives the first-order component of relative asset supply.

Section 9 discusses the zero-order components of the individual portfolio shares. Section 10 discusses Euler equation errors (Section 5.3 of the paper). Section 11 discusses balance of payment accounting, which is relevant for the companion paper "Capital Flows Under Dispersed Private Information" about the impact of private information on international capital flows. Section 12 discusses the model where the private signals are replaced by public signals.

2 The system of Equations

Denoting logs by lower case letters, the model is summarized by the consumption rule for young agents, the Tobin's Q investment rule, the portfolio Euler equations, the clearing of Home and Foreign equity markets, the dynamics of capital stocks, and the process of productivity, for $i = H,\ F$:

$$e^{c^i_{y,t}} = \frac{\omega}{1+\beta} e^{a_{i,t}+(1-\omega)k_{i,t}} \tag{1}$$

$$e^{i_{i,t}-k_{i,t}} = \delta + \frac{1}{\xi}\left(e^{q_{i,t}} - 1\right) \tag{2}$$

$$E^{H,j}_t e^{-r^{p,Hj}_{t+1}} \left(e^{r_{H,t+1}} - e^{r_{F,t+1}-\tau_{Hj,t}}\right) = 0 \tag{3}$$

$$E^{F,j}_t e^{-r^{p,Fj}_{t+1}} \left(e^{r_{H,t+1}-\tau_{Fj,t}} - e^{r_{F,t+1}}\right) = 0 \tag{4}$$

$$e^{q_{H,t}+k_{H,t+1}} = \frac{\beta\omega}{1+\beta}\left(z_{H,t}e^{a_{H,t}+(1-\omega)k_{H,t}} + z_{F,t}e^{a_{F,t}+(1-\omega)k_{F,t}}\right) \tag{5}$$

$$e^{q_{F,t}+k_{F,t+1}} = \frac{\beta\omega}{1+\beta}\left((1-z_{H,t})e^{a_{H,t}+(1-\omega)k_{H,t}} + (1-z_{F,t})e^{a_{F,t}+(1-\omega)k_{F,t}}\right) \tag{6}$$

$$e^{k_{i,t+1}} = (1-\delta)e^{k_{i,t}} + e^{i_{i,t}} \tag{7}$$

$$a_{i,t+1} = \rho a_{i,t} + \varepsilon_{i,t+1} \tag{8}$$

where the rates of returns on portfolios and equity are:

$$e^{r^{p,Hj}_{t+1}} = z_{Hj,t}e^{r_{H,t+1}} + (1-z_{Hj,t})e^{r_{F,t+1}-\tau_{Hj,t}} \tag{9}$$

$$e^{r^{p,Fj}_{t+1}} = z_{Fj,t}e^{r_{H,t+1}-\tau_{Fj,t}} + (1-z_{Fj,t})e^{r_{F,t+1}} \tag{10}$$

$$e^{r_{i,t+1}} = (1-\omega)e^{a_{i,t+1}-\omega k_{i,t+1}-q_{i,t}} + (1-\delta)e^{q_{i,t+1}-q_{i,t}} \tag{11}$$

We can compact this system of equations in two ways. First, we can remove the consumption equations. (1) gives a log-linear solution of consumption of young agents as a function of wealth:

$$c^i_{y,t} = ln(\omega/(1+\beta)) + a^i_t + (1-\omega)k^i_t \quad i = H, F \tag{12}$$

Consumption of young agents does not show up anywhere else in the system, so we can remove this control variable. Second, we combine (2) and (7) to eliminate investment $i_{i,t}$:

$$e^{k_{i,t+1}-k_{i,t}} = 1 + \frac{1}{\xi}\left(e^{q_{i,t}} - 1\right) \tag{13}$$

This reduces the number of control variables that remain to be solved to six: Home and foreign portfolio shares, equity prices and next period's capital stocks. The system of equations is then summarized by (3)-(6), (8) and (13). Once the resulting system is solved, we can compute the various order components of investment directly from (2).

We solve the equations in terms of worldwide averages of the variables and differences across countries. The variables for Home and Foreign countries can be expressed as functions of the worldwide average $x^A = 0.5\left(x^H + x^F\right)$ and cross-country difference $x^D = x^H - x^F$ as follows: $x^H = x^A + 0.5x^D$, $x^F = x^A - 0.5x^D$.

3 Taylor Expansions of Equations Other than Portfolio Eulers

We need to take cubic Taylor expansions of the portfolio Euler equations (3)-(4) and quadratic Taylor expansions of all other equations. In this section we derive quadratic Taylor expansions for all equations other than the portfolio Euler equations. In the next section we derive cubic expansions of the portfolio Euler equations.

3.1 Zero-Order Components

All expansions are done around the zero-order components of the variables, which are obtained by imposing the zero-order component of the equations:

$$a^A(0) = a^D(0) = q^A(0) = q^D(0) = k^D(0) = 0 \tag{14}$$

$$z^A(0) = 0.5 \tag{15}$$

$$k^A(0) = \frac{1}{\omega} ln(\beta\omega/(1+\beta)) \tag{16}$$

In addition we have

$$e^{r(0)} = 1 - \delta + (1-\omega)(1+\beta)/(\beta\omega) \tag{17}$$

where $r(0)$ is the zero order component of $r_{H,t+1}$, $r_{F,t+1}$, as well as the portfolio returns $r_{t+1}^{p,Hj}$ and $r_{t+1}^{p,Fj}$. The only variable whose zero-order component cannot be computed from the zero-order component of the equations is z_t^D. This will be computed later on from the second-order component of the difference in portfolio Euler equations. We will start by expanding all equations other than the portfolio Euler equations, which are more complex.

3.2 Quadratic Taylor Expansions

From here on all variables will be in deviation from their zero-order components. So for example $k_{i,t}$ now stands for the log of the capital stock minus $k(0)$.

Starting with the capital accumulation equations (13), we take the log of both sides and then a quadratic Taylor expansion of the expression on the right hand side. This gives

$$k_{i,t+1} - k_{i,t} = \frac{1}{\xi} q_{i,t} + \frac{1}{2} \frac{\xi - 1}{\xi^2} (q_{i,t})^2$$

Taking the average and difference between the Home and Foreign relations we get:

$$k^A_{t+1} - k^A_t = \frac{1}{\xi} q^A_t + \frac{1}{2} \frac{\xi - 1}{\xi^2} \left(\left(q^A_t\right)^2 + \frac{1}{4} \left(q^D_t\right)^2 \right) \tag{18}$$

$$k^D_{t+1} - k^D_t = \frac{1}{\xi} q^D_t + \frac{\xi - 1}{\xi^2} q^A_t q^D_t \tag{19}$$

Quadratic Taylor expansions of the asset market clearing equations (5)-(6) are

$$\begin{aligned} q_{H,t} + k_{H,t+1} = {} & \left(a^A_t + (1-\omega) k^A_t\right) + \frac{z^D(0)}{2} \left(a^D_t + (1-\omega) k^D_t\right) + 2 z^A_t \\ & + \frac{1 - \left(z^D(0)\right)^2}{8} \left(a^D_t + (1-\omega) k^D_t\right)^2 \\ & + \frac{1}{2} z^D_t \left(a^D_t + (1-\omega) k^D_t\right) - 2 \left(z^A_t\right)^2 \\ & - z^D(0) z^A_t \left(a^D_t + (1-\omega) k^D_t\right) \end{aligned}$$

and:

$$\begin{aligned} q_{F,t} + k_{F,t+1} = {} & \left(a^A_t + (1-\omega) k^A_t\right) - \frac{z^D(0)}{2} \left(a^D_t + (1-\omega) k^D_t\right) - 2 z^A_t \\ & + \frac{1 - \left(z^D(0)\right)^2}{8} \left(a^D_t + (1-\omega) k^D_t\right)^2 \\ & - \frac{1}{2} z^D_t \left(a^D_t + (1-\omega) k^D_t\right) - 2 \left(z^A_t\right)^2 \\ & - z^D(0) z^A_t \left(a^D_t + (1-\omega) k^D_t\right) \end{aligned}$$

Combining these we have

$$\begin{aligned} q^A_t + k^A_{t+1} = {} & a^A_t + (1-\omega) k^A_t - 2 \left(z^A_t\right)^2 \\ & + \frac{1 - \left(z^D(0)\right)^2}{8} \left(a^D_t + (1-\omega) k^D_t\right)^2 \\ & - z^D(0) z^A_t \left(a^D_t + (1-\omega) k^D_t\right) \end{aligned} \tag{20}$$

and:

$$
\begin{aligned}
q_t^D + k_{t+1}^D &= z^D(0)\left(a_t^D + (1-\omega)k_t^D\right) + 4z_t^A \\
&\quad + z_t^D\left(a_t^D + (1-\omega)k_t^D\right)
\end{aligned}
\tag{21}
$$

The shock process (8) does not need to be expanded as it is already linear. In terms of averages and differences it is

$$
a_{t+1}^A = \rho a_t^A + \varepsilon_{t+1}^A \quad ; \quad a_{t+1}^D = \rho a_t^D + \varepsilon_{t+1}^D \tag{22}
$$

4 Cubic Taylor Expansions of Portfolio Euler Equations

We now turn to the portfolio Euler conditions (3)-(4). We first write them by replacing $r_{H,t+1}$ and $r_{F,t+1}$ by $r_{t+1}^A + 0.5er_{t+1}$ and $r_{t+1}^A - 0.5er_{t+1}$ respectively, where $er_{t+1} = r_{H,t+1} - r_{F,t+1}$ is the excess return. This gives

$$
\begin{aligned}
0 &= E_t^{H,j} e^{-r_{t+1}^{p,Hj}}\left(e^{r_{t+1}^A + 0.5er_{t+1}} - e^{r_{t+1}^A - 0.5er_{t+1} - \tau_{Hj,t}}\right) \\
0 &= E_t^{F,j} e^{-r_{t+1}^{p,Fj}}\left(e^{r_{t+1}^A + 0.5er_{t+1} - \tau_{Fj,t}} - e^{r_{t+1}^A - 0.5er_{t+1}}\right)
\end{aligned}
$$

We take a cubic approximations of these relations around the zero-order allocation, where all rates of returns are identical and the friction is zero:

$$
\begin{aligned}
0 &= E_t^{H,j}(er_{t+1}) + \tau_{Hj,t} \qquad (23) \\
&\quad -\frac{1}{2}\tau_{Hj,t}E_t^{H,j}er_{t+1} + E_t^{H,j}\left(r_{t+1}^A - r_{t+1}^{p,Hj}\right)(er_{t+1} + \tau_{Hj,t}) \\
&\quad +\frac{1}{24}E_t^{H,j}(er_{t+1})^3 + \frac{1}{2}E_t^{H,j}\left(r_{t+1}^A - r_{t+1}^{p,Hj}\right)^2 er_{t+1} \\
0 &= E_t^{F,j}(er_{t+1}) - \tau_{Fj,t} \qquad (24) \\
&\quad -\frac{1}{2}\tau_{Fj,t}E_t^{F,j}er_{t+1} + E_t^{F,j}\left(r_{t+1}^A - r_{t+1}^{p,Fj}\right)(er_{t+1} - \tau_{Fj,t}) \\
&\quad +\frac{1}{24}E_t^{F,j}(er_{t+1})^3 + \frac{1}{2}E_t^{F,j}\left(r_{t+1}^A - r_{t+1}^{p,Fj}\right)^2 er_{t+1}
\end{aligned}
$$

In computing these relations, we used the fact that $\tau_{Hj,t}$ and $\tau_{Fj,t}$ have terms of order two and above. As we are only interested in terms up to order three, we can drop terms such as $(\tau_{Hj,t})^2$, or $(\tau_{Hj,t})^3$, or $er_{t+1}(\tau_{Hj,t})^2$, or $(er_{t+1})^2\tau_{Hj,t}$.

The last step is to express $r_{t+1}^{p,Hj}$ and $r_{t+1}^{p,Fj}$ as functions of the frictions and the rates of return on equity. Since our aim is to obtain a cubic approximation, (23)-(23) imply that it is sufficient to use a quadratic Taylor expansion for the portfolio returns. For the Home portfolio return, we take the log of (9) and then obtain a quadratic expansion of the right-hand side:

$$\begin{aligned} r_{t+1}^{p,Hj} &= r_{t+1}^{A} + \left(z_{Hj}(0) - \frac{1}{2}\right) er_{t+1} - (1 - z_{Hj}(0))\tau_{Hj,t} \\ &+ \frac{z_{Hj}(0)(1 - z_{Hj}(0))}{2}(er_{t+1} + \tau_{Hj,t})^2 + z_{Hj,t}(er_{t+1} + \tau_{Hj,t}) \end{aligned}$$

The quadratic Taylor expansion of the Foreign investor j's portfolio return (10) is:

$$\begin{aligned} r_{t+1}^{p,Fj} &= r_{t+1}^{A} + \left(z_{Fj}(0) - \frac{1}{2}\right) er_{t+1} - z_{Fj}(0)\tau_{Fj,t} \\ &+ \frac{z_{Fj}(0)(1 - z_{Fj}(0))}{2}(er_{t+1} - \tau_{Fj,t})^2 + z_{Fj,t}(er_{t+1} - \tau_{Fj,t}) \end{aligned}$$

Focusing on terms up to order two, which we saw is sufficient for the cubic expansion of the portfolio Euler equations, we get:

$$\begin{aligned} r_{t+1}^{p,Hj} &= r_{t+1}^{A} + \left(z_{Hj}(0) - \frac{1}{2}\right) er_{t+1} - (1 - z_{Hj}(0))\tau_{Hj,t} \\ &+ \frac{z_{Hj}(0)(1 - z_{Hj}(0))}{2}(er_{t+1})^2 + z_{Hj,t}er_{t+1} \\ r_{t+1}^{p,Fj} &= r_{t+1}^{A} + \left(z_{Fj}(0) - \frac{1}{2}\right) er_{t+1} - z_{Fj}(0)\tau_{Fj,t} \\ &+ \frac{z_{Fj}(0)(1 - z_{Fj}(0))}{2}(er_{t+1})^2 + z_{Fj,t}er_{t+1} \end{aligned}$$

Using our results, (23)-(24) become (up to cubic terms):

$$\begin{aligned} 0 &= E_t^{H,j}(er_{t+1}) + \tau_{Hj,t} - \frac{2z_{Hj}(0) - 1}{2}E_t^{H,j}(er_{t+1})^2 \\ &-(2z_{Hj}(0) - 1)\tau_{Hj,t}E_t^{H,j}er_{t+1} - z_{Hj,t}E_t^{H,j}(er_{t+1})^2 \\ &+ \left[\frac{1}{6} - z_{Hj}(0)(1 - z_{Hj}(0))\right]E_t^{H,j}(er_{t+1})^3 \end{aligned} \tag{25}$$

$$\begin{aligned} 0 &= E_t^{F,j}(er_{t+1}) - \tau_{Fj,t} - \frac{2z_{Fj}(0) - 1}{2}E_t^{F,j}(er_{t+1})^2 \\ &+(2z_{Fj}(0) - 1)\tau_{Fj,t}E_t^{F,j}er_{t+1} - z_{Fj,t}E_t^{F,j}(er_{t+1})^2 \\ &+ \left[\frac{1}{6} - z_{Fj}(0)(1 - z_{Fj}(0))\right]E_t^{F,j}(er_{t+1})^3 \end{aligned} \tag{26}$$

Next, we aggregate (25)-(26) across investors in each country:

$$\begin{aligned}
0 &= \bar{E}_t^H er_{t+1} + \tau_t^H - \int_0^1 \frac{2z_{Hj}(0)-1}{2} E_t^{H,j}(er_{t+1})^2 dj \\
&\quad - \int_0^1 (2z_{Hj}(0)-1)\tau_{Hj,t} E_t^{H,j} er_{t+1} dj - \int_0^1 z_{Hj,t} E_t^{H,j}(er_{t+1})^2 dj \\
&\quad + \int_0^1 \left[\frac{1}{6} - z_{Hj}(0)(1-z_{Hj}(0))\right] E_t^{H,j}(er_{t+1})^3 dj \\
0 &= \bar{E}_t^F er_{t+1} - \tau_t^F - \int_0^1 \frac{2z_{Fj}(0)-1}{2} E_t^{F,j}(er_{t+1})^2 dj \\
&\quad + \int_0^1 (2z_{Fj}(0)-1)\tau_{Fj,t} E_t^{F,j} er_{t+1} dj - \int_0^1 z_{Fj,t} E_t^{F,j}(er_{t+1})^2 dj \\
&\quad + \int_0^1 \left[\frac{1}{6} - z_{Fj}(0)(1-z_{Fj}(0))\right] E_t^{F,j}(er_{t+1})^3 dj
\end{aligned}$$

where $\bar{E}_t^H er_{t+1} = \int_0^1 E_t^{Hj} dj$ is the average expectation of excess return across Home agents, and $\bar{E}_t^F er_{t+1} = \int_0^1 E_t^{Fj} dj$ is the average expectation of Foreign agents.

We will show that the expected value of $(er_{t+1})^2$ and $(er_{t+1})^3$ is the same across all agents to all relevant orders. The portfolio Euler equations, averaged across agents, are then:

$$\begin{aligned}
0 &= \bar{E}_t^H er_{t+1} + \tau_t^H - \left(z^A(0) - \frac{1}{2} + \frac{z^D(0)}{2}\right) E_t (er_{t+1})^2 \\
&\quad + 2\tau e_t^H - \left(z_t^A + \frac{1}{2} z_t^D\right) E_t (er_{t+1})^2 + m(0) E_t (er_{t+1})^3 \\
0 &= \bar{E}_t^F er_{t+1} - \tau_t^F - \left(z^A(0) - \frac{1}{2} - \frac{z^D(0)}{2}\right) E_t (er_{t+1})^2 \\
&\quad - 2\tau e_t^F - \left(z_t^A - \frac{1}{2} z_t^D\right) E_t (er_{t+1})^2 + m(0) E_t (er_{t+1})^3
\end{aligned}$$

where we used the fact that the component of $\tau_{Hj,t}$ that enters $\tau_{Hj,t} E_t^{H,j} er_{t+1}$ is only the second-order component τ, as we focus on terms up to a third-order, and defined (note that by symmetry $\int_0^1 z_{Hj}(0))(1 - z_{Hj}(0))dj = \int_0^1 z_{Fj}(0))(1 -$

$z_{Fj}(0))dj$):

$$\begin{aligned} m(0) &= \frac{1}{6} - \int_0^1 z_{Hj}(0)(1 - z_{Hj}(0))dj \\ e_t^H &= \int_0^1 (0.5 - z_{Hj}(0)) E_t^{Hj} er_{t+1} dj \\ e_t^F &= \int_0^1 (0.5 - z_{Fj}(0)) E_t^{Fj} er_{t+1} dj \end{aligned}$$

Taking the average and difference of the relations, we get:

$$\begin{aligned} \left(z_t^A + z^A(0) - \frac{1}{2}\right) E_t (er_{t+1})^2 &= \bar{E}_{t+1}^A er_{t+1} + \frac{\tau_t^D}{2} + \tau e_t^D \\ &\quad + m(0) E_t (er_{t+1})^3 \end{aligned} \tag{27}$$

$$\left(z_t^D + z^D(0)\right) E_t (er_{t+1})^2 = \bar{E}_t^H er_{t+1} - \bar{E}_t^F er_{t+1} + 2\tau + 4\tau e_t^A \tag{28}$$

where $\bar{E}_{t+1}^A er_{t+1} = 0.5 \left(\bar{E}_t^H er_{t+1} + \bar{E}_t^F er_{t+1}\right)$.

5 Deriving Expression for Excess Return

The expansions of the portfolio Euler equations depend on er_{t+1}, er_{t+1}^2 and er_{t+1}^3, for which we need cubic expansions. We need in particular to express these variables in terms of variables known at time t and future innovations. This will allow us to compute expectations later on, once we have solved a signal extraction problem regarding future innovations.

5.1 Cubic Taylor Expansion of Excess Return

We start by taking the log of (11) as:

$$r_{i,t+1} = \ln\left[(1-\omega) e^{x_{t+1}^i} + (1-\delta)\right] + q_{i,t+1} - q_{i,t}$$

where $x_{t+1}^i = a_{i,t+1} - \omega k_{i,t+1} - q_{i,t+1}$. A cubic expansion of this relation implies:

$$r_{i,t+1} = q_{i,t+1} - q_{i,t} + \delta_1 x_{t+1}^i + \frac{1}{2}\delta_2 \left(x_{t+1}^i\right)^2 + \frac{1}{6}\delta_3 \left(x_{t+1}^i\right)^3$$

where $r_{i,t+1}$ is in deviation from $r(0)$ and:

$$\begin{aligned} \delta_1 &= \frac{(1-\omega)(1+\beta)}{(1-\omega)(1+\beta) + (1-\delta)\beta\omega} \\ \delta_2 &= \delta_1 - (\delta_1)^2 \\ \delta_3 &= \delta_2 - 2\delta_1\delta_2 \end{aligned}$$

Combining the cubic approximation for $r_{H,t+1}$ and $r_{F,t+1}$ we write the cubic approximation of the excess return as:

$$\begin{aligned}
er_{t+1} &= q_{t+1}^D - q_t^D + \delta_1 x_{t+1}^D + \delta_2 x_{t+1}^A x_{t+1}^D + \delta_3 \left[\frac{1}{24}\left(x_{t+1}^D\right)^3 + \frac{1}{2}\left(x_{t+1}^A\right)^2 x_{t+1}^D\right] \\
&= (1-\delta_1) q_{t+1}^D - q_t^D + \delta_1\left(a_{t+1}^D - \omega k_{t+1}^D\right) \\
&\quad +\delta_2\left(a_{t+1}^A - \omega k_{t+1}^A - q_{t+1}^A\right)\left(a_{t+1}^D - \omega k_{t+1}^D - q_{t+1}^D\right) \\
&\quad +\frac{\delta_3}{24}\left(a_{t+1}^D - \omega k_{t+1}^D - q_{t+1}^D\right)^3 + \frac{\delta_3}{2}\left(a_{t+1}^A - \omega k_{t+1}^A - q_{t+1}^A\right)^2\left(a_{t+1}^D - \omega k_{t+1}^D - q_{t+1}^D\right)
\end{aligned} \tag{29}$$

Using this, cubic expressions for er_{t+1}^2 and er_{t+1}^3 are:

$$\begin{aligned}
er_{t+1}^2 &= \left[(1-\delta_1) q_{t+1}^D - q_t^D + \delta_1\left(a_{t+1}^D - \omega k_{t+1}^D\right)\right]^2 \\
&\quad +2\delta_2\begin{bmatrix} (1-\delta_1) q_{t+1}^D - q_t^D \\ +\delta_1\left(a_{t+1}^D - \omega k_{t+1}^D\right)\end{bmatrix}\left(a_{t+1}^A - \omega k_{t+1}^A - q_{t+1}^A\right)\left(a_{t+1}^D - \omega k_{t+1}^D - q_{t+1}^D\right) \\
er_{t+1}^3 &= \left[(1-\delta_1) q_{t+1}^D - q_t^D + \delta_1\left(a_{t+1}^D - \omega k_{t+1}^D\right)\right]^3
\end{aligned}$$

5.2 Asset Price Conjectures

These cubic Taylor expansions for the excess return depend on future asset price innovations. We need to make a conjecture for the asset prices as a function of the state variables, so that we can split up q_{t+1}^D and q_{t+1}^A into components that are known at time t and future innovations. Our conjectures are

$$q_t^D = \alpha_{qD} S_t + \alpha_{5,qD} h_t + S_t' A_{qD} S_t + \beta_{qD} S_t h_t + \mu_{qD}\left(h_t\right)^2 + cubic_D(S_t, h_t) \tag{30}$$

$$q_t^A = \alpha_{qA} S_t + \alpha_{5,qA} h_t + S_t' A_{qA} S_t + \beta_{qA} S_t h_t + \mu_{qA}\left(h_t\right)^2 \tag{31}$$

where $S_t = (a_t^D, a_t^A, k_t^D, k_t^A)'$ is the vector or observed state variables, and $h_t = \varepsilon_{t+1}^D + \lambda \tau_t^D / \tau$ is the combination of unobserved state variables (future payoff innovation and current noise shock). λ is a zero-order term that needs to be solved. We refer to it as the noise to signal ratio in the public signal.

$\alpha_{qD} = (\alpha_{1,qD}, \alpha_{2,qD}, \alpha_{3,qD}, \alpha_{4,qD})'$, $\alpha_{qA} = (\alpha_{1,qA}, \alpha_{2,qA}, \alpha_{3,qA}, \alpha_{4,qA})'$ are vectors of coefficients and A_{qD}, A_{qA} are symmetric 4 by 4 matrices. $\beta_{qD} = (\beta_{1,qD}, \beta_{2,qD}, \beta_{3,qD}, \beta_{4,qD})'$ and $\beta_{qA} = (\beta_{1,qA}, \beta_{2,qA}, \beta_{3,qA}, \beta_{4,qA})'$.

$cubic_D(S_t, h_t)$ stands for all 35 cubic terms in the elements of S_t and h_t in the relative asset price. We only need to allow for cubic terms in the expression for q_t^D as it affects the excess return er_{t+1} linearly. By contrast, the average asset price q_t^A

only enters the excess return interacted with other variables, making a quadratic conjecture sufficient to obtain a cubic expression for er_{t+1}. As we will see, it is not essential for the solution to explicitly write out the sum of all the 35 cubic terms in $cubic_D(S_t, h_t)$.

It is useful to write down the resulting expressions for q^D_{t+1} and q^A_{t+1} that enter in the expression for er_{t+1}. Define $\bar{S}_{t+1} = (\rho a^D_t, \rho a^A_t, k^D_{t+1}, k^A_{t+1})'$. This is the component of S_{t+1} that is known at time t. We then have

$$S_{t+1} = \bar{S}_{t+1} + \begin{pmatrix} \varepsilon^D_{t+1} \\ \varepsilon^A_{t+1} \\ 0 \\ 0 \end{pmatrix} \tag{32}$$

Using this, (30) at time $t+1$ is:

$$\begin{aligned} q^D_{t+1} \;=\; & \alpha_{qD}\bar{S}_{t+1} + \alpha_{1,qD}\varepsilon^D_{t+1} + \alpha_{2,qD}\varepsilon^A_{t+1} + \alpha_{5,qD}h_{t+1} \\ & +\bar{S}'_{t+1}A_{qD}\bar{S}_{t+1} + 2\begin{pmatrix} \varepsilon^D_{t+1} & \varepsilon^A_{t+1} & 0 & 0 \end{pmatrix} A_{qD}\bar{S}_{t+1} \\ & +A_{11,qD}\left(\varepsilon^D_{t+1}\right)^2 + 2A_{12,qD}\varepsilon^A_{t+1}\varepsilon^D_{t+1} + A_{22,qD}\left(\varepsilon^A_{t+1}\right)^2 \\ & +\beta_{qD}\bar{S}_{t+1}h_{t+1} + \left(\beta_{1,qD}\varepsilon^D_{t+1} + \beta_{2,qD}\varepsilon^A_{t+1}\right)h_{t+1} + \mu_{qD}\left(h_{t+1}\right)^2 \\ & +cubic_D(a^D_t, a^A_t, k^D_{t+1}, k^A_{t+1}, h_{t+1}, \varepsilon^D_{t+1}, \varepsilon^A_{t+1}) \end{aligned} \tag{33}$$

where $\alpha_{i,qD}$ os the ith element of the vector α_{qD}, and $A_{ij,qD}$ is the element on the ith row and jth column of A_{qD} ($A_{12,qD} = A_{21,qD}$ by symmetry). All we need to know about the cubic terms in the last row is that they can be written as terms that are cubic in a^D_t, a^A_t, k^D_{t+1}, k^A_{t+1}, h_{t+1}, ε^D_{t+1}, ε^A_{t+1}, with the last three variables unknown at time t.

Similarly, we write (31) at time $t+1$ as:

$$\begin{aligned} q^A_{t+1} \;=\; & \alpha_{qA}\bar{S}_{t+1} + \alpha_{1,qA}\varepsilon^D_{t+1} + \alpha_{2,qA}\varepsilon^A_{t+1} \\ & +\bar{S}'_{t+1}A_{qA}\bar{S}_{t+1} + 2\begin{pmatrix} \varepsilon^D_{t+1} & \varepsilon^A_{t+1} & 0 & 0 \end{pmatrix} A_{qA}\bar{S}_{t+1} \\ & +A_{11,qA}\left(\varepsilon^D_{t+1}\right)^2 + 2A_{12,qA}\varepsilon^A_{t+1}\varepsilon^D_{t+1} + A_{22,qA}\left(\varepsilon^A_{t+1}\right)^2 \\ & +\beta_{qA}\bar{S}_{t+1}h_{t+1} + \left(\beta_{1,qA}\varepsilon^D_{t+1} + \beta_{2,qA}\varepsilon^A_{t+1}\right)h_{t+1} + \mu_{qA}\left(h_{t+1}\right)^2 \end{aligned} \tag{34}$$

5.3 Cubic Excess Return Expressions in Knowns and Un-Knowns

We now use our conjectures on the asset price to write a cubic expression for the excess return. It follows from (29) that a cubic expression for er_{t+1} requires a quadratic expression for $a^A_{t+1} - \omega k^A_{t+1} - q^A_{t+1}$ and cubic expressions for $a^D_{t+1} - \omega k^D_{t+1} - q^D_{t+1}$ and $(1-\delta_1)q^D_{t+1} - q^D_t + \delta_1 \left(a^D_{t+1} - \omega k^D_{t+1}\right)$.

A quadratic expression for $a^A_{t+1} - \omega k^A_{t+1} - q^A_{t+1}$ is

$$
\begin{aligned}
& a^A_{t+1} - \omega k^A_{t+1} - q^A_{t+1} \\
= \ & \left(I_2 - \omega I_4 - \alpha_{qA}\right) \bar{S}_{t+1} + \left(1 - \alpha_{2,qA}\right) \varepsilon^A_{t+1} - \alpha_{1,qA} \varepsilon^D_{t+1} \\
- & \left[\begin{array}{c} \bar{S}'_{t+1} A_{qA} \bar{S}_{t+1} + 2 \begin{pmatrix} \varepsilon^D_{t+1} & \varepsilon^A_{t+1} & 0 & 0 \end{pmatrix} A_{qA} \bar{S}_{t+1} \\ + A_{11,qA} \left(\varepsilon^D_{t+1}\right)^2 + 2A_{12,qA} \varepsilon^A_{t+1} \varepsilon^D_{t+1} + A_{22,qA} \left(\varepsilon^A_{t+1}\right)^2 \\ + \beta_{qA} \bar{S}_{t+1} h_{t+1} + \left(\beta_{1,qA} \varepsilon^D_{t+1} + \beta_{2,qA} \varepsilon^A_{t+1}\right) h_{t+1} + \mu_{qA} \left(h_{t+1}\right)^2 \end{array} \right]
\end{aligned}
$$

where I_i is a 1x4 vector of zeros except for the ith element equal to one.

A cubic expression for $a^D_{t+1} - \omega k^D_{t+1} - q^D_{t+1}$ is

$$
\begin{aligned}
& a^D_{t+1} - \omega k^D_{t+1} - q^D_{t+1} \\
= \ & \left(I_1 - \omega I_3 - \alpha_{qD}\right) \bar{S}_{t+1} + \left(1 - \alpha_{1,qD}\right) \varepsilon^D_{t+1} - \alpha_{2,qD} \varepsilon^A_{t+1} - \alpha_{5,qD} h_{t+1} \\
- & \left[\begin{array}{c} \bar{S}'_{t+1} A_{qD} \bar{S}_{t+1} + 2 \begin{pmatrix} \varepsilon^D_{t+1} & \varepsilon^A_{t+1} & 0 & 0 \end{pmatrix} A_{qD} \bar{S}_{t+1} \\ + A_{11,qD} \left(\varepsilon^D_{t+1}\right)^2 + 2A_{12,qD} \varepsilon^A_{t+1} \varepsilon^D_{t+1} + A_{22,qD} \left(\varepsilon^A_{t+1}\right)^2 \\ + \beta_{qD} \bar{S}_{t+1} h_{t+1} + \left(\beta_{1,qD} \varepsilon^D_{t+1} + \beta_{2,qD} \varepsilon^A_{t+1}\right) h_{t+1} + \mu_{qD} \left(h_{t+1}\right)^2 \\ + cubic_D(a^D_t, a^A_t, k^D_{t+1}, k^A_{t+1}, h_{t+1}, \varepsilon^D_{t+1}, \varepsilon^A_{t+1}) \end{array} \right]
\end{aligned}
$$

Finally, a cubic expression for $(1-\delta_1)q^D_{t+1} - q^D_t + \delta_1 \left(a^D_{t+1} - \omega k^D_{t+1}\right)$ is

$$
\begin{aligned}
& (1-\delta_1)q^D_{t+1} - q^D_t + \delta_1 \left(a^D_{t+1} - \omega k^D_{t+1}\right) \\
= \ & \left[(1-\delta_1)\alpha_{qD} + \delta_1 \left(I_1 - \omega I_3\right)\right] \bar{S}_{t+1} - \alpha_{qD} S_t \\
& + \left[(1-\delta_1)\alpha_{1,qD} + \delta_1\right] \varepsilon^D_{t+1} + (1-\delta_1)\alpha_{2,qD} \varepsilon^A_{t+1} \\
& + \alpha_{5,qD} \left[(1-\delta_1) h_{t+1} - h_t\right] \\
& + (1-\delta_1) \left[\begin{array}{c} \bar{S}'_{t+1} A_{qD} \bar{S}_{t+1} + 2 \begin{pmatrix} \varepsilon^D_{t+1} & \varepsilon^A_{t+1} & 0 & 0 \end{pmatrix} A_{qD} \bar{S}_{t+1} \\ + A_{11,qD} \left(\varepsilon^D_{t+1}\right)^2 + 2A_{12,qD} \varepsilon^A_{t+1} \varepsilon^D_{t+1} + A_{22,qD} \left(\varepsilon^A_{t+1}\right)^2 \\ + \beta_{qD} \bar{S}_{t+1} h_{t+1} + \left(\beta_{1,qD} \varepsilon^D_{t+1} + \beta_{2,qD} \varepsilon^A_{t+1}\right) h_{t+1} + \mu_{qD} \left(h_{t+1}\right)^2 \\ + cubic_D(a^D_t, a^A_t, k^D_{t+1}, k^A_{t+1}, h_{t+1}, \varepsilon^D_{t+1}, \varepsilon^A_{t+1}) \end{array} \right] \\
& - \left[S'_t A_{qD} S_t + \beta_{qD} S_t h_t + \mu_{qD} \left(h_t\right)^2 + cubic_D(S_t, h_t)\right]
\end{aligned}
$$

It is useful to distinguish between linear and quadratic terms in er_{t+1} that are known at time t, denoted by lin_t and $quad_t$. Substituting the results above into (29), the cubic expression for er_{t+1} is

$$\begin{aligned} er_{t+1} &= lin_t + quad_t + \mu_{er,1}\varepsilon^D_{t+1} + \mu_{er,2}\varepsilon^A_{t+1} + \mu_{er,3}h_{t+1} \\ &+\mu_{er,4}(\varepsilon^D_{t+1})^2 + \mu_{er,5}(\varepsilon^A_{t+1})^2 + \mu_{er,6}\varepsilon^D_{t+1}\varepsilon^A_{t+1} \\ &+\mu_{er,7}\varepsilon^D_{t+1}h_{t+1} + \mu_{er,8}\varepsilon^A_{t+1}h_{t+1} + (1-\delta_1)\mu_{qD}h^2_{t+1} \\ &+cubic_{er}(S_t, k^D_{t+1}, k^A_{t+1}, h_t, h_{t+1}, \varepsilon^D_{t+1}, \varepsilon^A_{t+1}) \end{aligned} \tag{35}$$

where:

$$\begin{aligned} lin_t &= \left[(1-\delta_1)\alpha_{qD} + \delta_1\left(I_1 - \omega I_3\right)\right]\bar{S}_{t+1} - \alpha_{qD}S_t - \alpha_{5,qD}h_t \\ quad_t &= (1-\delta_1)\bar{S}'_{t+1}A_{qD}\bar{S}_{t+1} - \left[S'_tA_{qD}S_t + \beta_{qD}S_th_t + \mu_{qD}\left(h_t\right)^2\right] \\ &+\delta_2\bar{S}'_{t+1}\left(I_1 - \omega I_3 - \alpha_{qD}\right)'\left(I_2 - \omega I_4 - \alpha_{qA}\right)\bar{S}_{t+1} \end{aligned}$$

and ($A_{qD,\text{row } i}$ is row i of A_{qD}):

$$\begin{aligned} \mu_{er,1} &= (1-\delta_1)\alpha_{1,qD} + \delta_1 + 2(1-\delta_1)A_{qD,\text{row }1}\bar{S}_{t+1} \\ &+\delta_2\left[(1-\alpha_{1,qD})\left(I_2 - \omega I_4 - \alpha_{qA}\right) - \alpha_{1,qA}\left(I_1 - \omega I_3 - \alpha_{qD}\right)\right]\bar{S}_{t+1} \\ \mu_{er,2} &= (1-\delta_1)\alpha_{2,qD} + 2(1-\delta_1)A_{qD,\text{row }2}\bar{S}_{t+1} \\ &-\delta_2\left[\alpha_{2,qD}\left(I_2 - \omega I_4 - \alpha_{qA}\right) - (1-\alpha_{2,qA})\left(I_1 - \omega I_3 - \alpha_{qD}\right)\right]\bar{S}_{t+1} \\ \mu_{er,3} &= \alpha_{5,qD}(1-\delta_1) + (1-\delta_1)\beta_{qD}\bar{S}_{t+1} \\ &-\delta_2\alpha_{5,qD}\left(I_2 - \omega I_4 - \alpha_{qA}\right)\bar{S}_{t+1} \\ \mu_{er,4} &= (1-\delta_1)A_{11,qD} - \delta_2\alpha_{1,qA}\left(1-\alpha_{1,qD}\right) \\ \mu_{er,5} &= (1-\delta_1)A_{22,qD} - \delta_2\alpha_{2,qD}\left(1-\alpha_{2,qA}\right) \\ \mu_{er,6} &= 2(1-\delta_1)A_{12,qD} + \delta_2\left((1-\alpha_{1,qD})(1-\alpha_{2,qA}) + \alpha_{2,qD}\alpha_{1,qA}\right) \\ \mu_{er,7} &= (1-\delta_1)\beta_{1,qD} + \delta_2\alpha_{1,qA}\alpha_{5,qD} \\ \mu_{er,8} &= (1-\delta_1)\beta_{2,qD} - \delta_2(1-\alpha_{2,qA})\alpha_{5,qD} \end{aligned}$$

The expressions for er^2_{t+1} and er^3_{t+1} are then:

$$\begin{aligned} er^2_{t+1} &= (lin_t)^2 + 2lin_t\left[\mu_{er,1}\varepsilon^D_{t+1} + \mu_{er,2}\varepsilon^A_{t+1} + \mu_{er,3}h_{t+1}\right] \\ &+\left(\mu_{er,1}\right)^2\left(\varepsilon^D_{t+1}\right)^2 + \left(\mu_{er,2}\right)^2\left(\varepsilon^A_{t+1}\right)^2 + 2\mu_{er,1}\mu_{er,2}\varepsilon^A_{t+1}\varepsilon^D_{t+1} \\ &+\left(\mu_{er,3}\right)^2\left(h_{t+1}\right)^2 + 2\mu_{er,3}\left[\mu_{er,1}\varepsilon^D_{t+1}h_{t+1} + \mu_{er,2}\varepsilon^A_{t+1}h_{t+1}\right] \\ &+cubic_{er2}(S_t, k^D_{t+1}, k^A_{t+1}, h_t, h_{t+1}, \varepsilon^D_{t+1}, \varepsilon^A_{t+1}) \end{aligned} \tag{36}$$

$$er^3_{t+1} = cubic_{er3}(S_t, k^D_{t+1}, k^A_{t+1}, h_t, h_{t+1}, \varepsilon^D_{t+1}, \varepsilon^A_{t+1}) \tag{37}$$

Here $cubic_{er2}(.)$ refers to terms of er_{t+1}^2 that are cubic in the state variables and innovations. Similarly $cubic_{er3}(.)$ refers to terms of er_{t+1}^3 that are cubic in the state variables and innovations. It corresponds to the cubic product of the linear terms in er_{t+1} in (35): $cubic_{er3}(.) = \left(lin_t + \mu_{er,1}\varepsilon_{t+1}^D + \mu_{er,2}\varepsilon_{t+1}^A + \mu_{er,3}h_{t+1}\right)^3$.

6 Signal extraction

6.1 General formula

Investor combine three sources of information on the future payoff innovations $\varepsilon_{H,t+1}$ and $\varepsilon_{F,t+1}$. The first is their unconditional distribution, with each having a mean zero and a variance σ_a^2. The second is the private signals, which for a Home investor are:

$$v_{j,t}^{H,H} = \varepsilon_{H,t+1} + \epsilon_{j,t}^{H,H} \qquad ; \qquad v_{j,t}^{H,F} = \varepsilon_{F,t+1} + \epsilon_{j,t}^{H,F}$$

The signals for a Foreign investor are:

$$v_{j,t}^{F,H} = \varepsilon_{H,t+1} + \epsilon_{j,t}^{F,H} \qquad ; \qquad v_{j,t}^{F,F} = \varepsilon_{F,t+1} + \epsilon_{j,t}^{F,F}$$

The variance of signal errors are $\sigma_{H,H}^2$ for signals on the domestic innovation, and $\sigma_{H,F}^2 > \sigma_{H,H}^2$ for innovations in the other country.

The final source of information stems from the asset price difference, more specifically:

$$h_t = \varepsilon_{t+1}^D + \lambda \frac{\tau_t^D}{\tau} \tag{38}$$

where λ is the noise to signal ratio, and the noise τ_t^D/τ is mean zero and variance $4\theta\sigma_a^2$.

A Home investor j has a vector of signals $Y_t^{j,H} = \left|h_t, v_{j,t}^{H,H}, v_{j,t}^{H,F}, 0, 0\right|'$. This is linked to the innovation vector $\xi_{t+1} = |\varepsilon_{H,t+1}, \varepsilon_{F,t+1}|'$ through a matrix $X^{j,H}$ with shocks $v^{j,H}$: $Y_t^{j,H} = X^{j,H}\xi_{t+1} + v_t^{j,H}$. The shocks $v_t^{j,H}$ have a diagonal variance matrix $R^{j,H}$. The expectation and variance of ξ_{t+1} are then:

$$\begin{aligned}
E_t^{j,H}\xi_{t+1} &= \left(\left(X^{j,H}\right)'\left(R^{j,H}\right)^{-1}\left(X^{j,H}\right)\right)^{-1}\left(X^{j,H}\right)'\left(R^{j,H}\right)^{-1}Y_t^{j,H} \\
V_t^{j,H}\left(\xi_{t+1}\right) &= \left(\left(X^{j,H}\right)'\left(R^{j,H}\right)^{-1}\left(X^{j,H}\right)\right)^{-1}
\end{aligned}$$

The formula is similar for a Foreign investor.

6.2 Home investor

The specific matrices for the Home investor j are:

$$X^{j,H} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \quad ; \quad diag\left(R^{j,H}\right) = \begin{vmatrix} 4\lambda^2\theta\sigma_a^2 \\ \sigma_{H,H}^2 \\ \sigma_{H,F}^2 \\ \sigma_a^2 \\ \sigma_a^2 \end{vmatrix}$$

The inferences for the Home investor are then:

$$E_t^{j,H}\xi_{t+1} = \begin{vmatrix} \alpha_{\varepsilon H,h}^{j,H} h_t + \alpha_{\varepsilon H,vH}^{j,H} v_{j,t}^{H,H} + \alpha_{\varepsilon H,vF}^{j,H} v_{j,t}^{H,F} \\ \alpha_{\varepsilon F,h}^{j,H} h_t + \alpha_{\varepsilon F,vH}^{j,H} v_{j,t}^{H,H} + \alpha_{\varepsilon F,vF}^{j,H} v_{j,t}^{H,F} \end{vmatrix} \tag{39}$$

$$V_t\left(\xi_{t+1}\right) = \frac{\sigma_a^2}{V} \begin{vmatrix} \frac{1}{4\lambda^2\theta} + \frac{\sigma_a^2}{\sigma_{H,F}^2} + 1 & \frac{1}{4\lambda^2\theta} \\ \frac{1}{4\lambda^2\theta} & \frac{1}{4\lambda^2\theta} + \frac{\sigma_a^2}{\sigma_{H,H}^2} + 1 \end{vmatrix} \tag{40}$$

where:

$$V = \left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right) + \frac{1}{4\lambda^2\theta}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + \frac{\sigma_a^2}{\sigma_{H,H}^2} + 2\right)$$

$$\alpha_{\varepsilon H,h}^{j,H} = \frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1\right)\frac{1}{4\lambda^2\theta}$$

$$\alpha_{\varepsilon H,vH}^{j,H} = \frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,F}^2} + 1 + \frac{1}{4\lambda^2\theta}\right)\frac{\sigma_a^2}{\sigma_{H,H}^2}$$

$$\alpha_{\varepsilon H,vF}^{j,H} = \frac{1}{V}\frac{1}{4\lambda^2\theta}\frac{\sigma_a^2}{\sigma_{H,F}^2}$$

$$\alpha_{\varepsilon F,h}^{j,H} = -\frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1\right)\frac{1}{4\lambda^2\theta}$$

$$\alpha_{\varepsilon F,vH}^{j,H} = \frac{1}{V}\frac{1}{4\lambda^2\theta}\frac{\sigma_a^2}{\sigma_{H,H}^2}$$

$$\alpha_{\varepsilon F,vF}^{j,H} = \frac{1}{V}\left(\frac{\sigma_a^2}{\sigma_{H,H}^2} + 1 + \frac{1}{4\lambda^2\theta}\right)\frac{\sigma_a^2}{\sigma_{H,F}^2}$$

6.3 Foreign investor

The specific matrices for the Foreign investor j are:

$$X^{j,F} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad ; \qquad diag\left(R^{j,F}\right) = \begin{vmatrix} 4\lambda^2\theta\sigma_a^2 \\ \sigma_{H,F}^2 \\ \sigma_{H,H}^2 \\ \sigma_a^2 \\ \sigma_a^2 \end{vmatrix}$$

The inferences for the Foreign investor are then:

$$E_t^{j,F}\xi_{t+1} = \begin{vmatrix} \alpha_{\varepsilon H,h}^{j,F} h_t + \alpha_{\varepsilon H,vH}^{j,F} v_{j,t}^{F,H} + \alpha_{\varepsilon H,vF}^{j,F} v_{j,t}^{F,F} \\ \alpha_{\varepsilon F,h}^{j,F} h_t + \alpha_{\varepsilon F,vH}^{j,F} v_{j,t}^{F,H} + \alpha_{\varepsilon F,vF}^{j,F} v_{j,t}^{F,F} \end{vmatrix} \tag{41}$$

$$V_t\left(\xi_{t+1}\right) = \frac{\sigma_a^2}{V} \begin{vmatrix} \frac{1}{4\lambda^2\theta} + \frac{\sigma_a^2}{\sigma_{H,F}^2} + 1 & \frac{1}{4\lambda^2\theta} \\ \frac{1}{4\lambda^2\theta} & \frac{1}{4\lambda^2\theta} + \frac{\sigma_a^2}{\sigma_{H,H}^2} + 1 \end{vmatrix} \tag{42}$$

where V is as before and:

$$\begin{aligned} \alpha_{\varepsilon H,h}^{j,F} &= -\alpha_{\varepsilon F,h}^{j,H} \\ \alpha_{\varepsilon H,vH}^{j,F} &= \alpha_{\varepsilon F,vF}^{j,H} \\ \alpha_{\varepsilon H,vF}^{j,F} &= \alpha_{\varepsilon F,vH}^{j,H} \\ \alpha_{\varepsilon F,h}^{j,F} &= -\alpha_{\varepsilon H,h}^{j,H} \\ \alpha_{\varepsilon F,vH}^{j,F} &= \alpha_{\varepsilon H,vF}^{j,H} \\ \alpha_{\varepsilon F,vF}^{j,F} &= \alpha_{\varepsilon H,vH}^{j,H} \end{aligned}$$

6.4 Orders of coefficients

The coefficients in (39)-(40) are functions of σ_a^2 and thus include various orders. The orders of a coefficient α are computed by taking a Taylor expansion with respect to σ_a around $\sigma_a = 0$:

$$\alpha = \alpha|_{\sigma_a=0} + \left.\frac{\partial\alpha}{\partial\sigma_a}\right|_{\sigma_a=0} \sigma_a + \frac{1}{2} \left.\frac{\partial^2\alpha}{\left(\partial\sigma_a\right)^2}\right|_{\sigma_a=0} \sigma_a^2 + \frac{1}{3} \left.\frac{\partial^3\alpha}{\left(\partial\sigma_a\right)^3}\right|_{\sigma_a=0} \sigma_a^3 + ...$$

where the terms on the right-hand side are the zero-, first-, second- and third-order components of the coefficient.

The only term in the variance (40) that is non-zero is the second-order term:

$$\left[V_t\left(\xi_{t+1}\right)\right](2)=\frac{1}{2\left(1+2\lambda^2\theta\right)}\left|\begin{array}{cc}1+4\lambda^2\theta & 1\\ 1 & 1+4\lambda^2\theta\end{array}\right|\sigma_a^2$$

The coefficients on the private signals in (39) only has second-order terms:

$$\begin{aligned}\left[\alpha_{\varepsilon H,vH}^{j,H}\right](2) &= \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma_{H,H}^2}\\ \left[\alpha_{\varepsilon H,vF}^{j,H}\right](2) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma_{H,F}^2}\\ \left[\alpha_{\varepsilon F,vH}^{j,H}\right](2) &= \frac{1}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma_{H,H}^2}\\ \left[\alpha_{\varepsilon F,vF}^{j,H}\right](2) &= \frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\frac{\sigma_a^2}{\sigma_{H,F}^2}\end{aligned}$$

The coefficients on the signal h_t in (39) have zero- and second-order terms:

$$\begin{aligned}\left[\alpha_{\varepsilon H,h}^{j,H}\right](0) &= -\left[\alpha_{\varepsilon F,h}^{j,H}\right](0)=\frac{1}{2\left(1+2\lambda^2\theta\right)}\\ \left[\alpha_{\varepsilon H,h}^{j,H}\right](2) &= \frac{\sigma_{H,H}^2-\left(1+4\lambda^2\theta\right)\sigma_{H,F}^2}{4\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}\sigma_a^2\\ \left[\alpha_{\varepsilon F,h}^{j,H}\right](2) &= \frac{\left(1+4\lambda^2\theta\right)\left(\sigma_{H,H}^2\right)-\sigma_{H,F}^2}{4\left(1+2\lambda^2\theta\right)^2\sigma_{H,H}^2\sigma_{H,F}^2}\sigma_a^2\end{aligned}$$

The signal h_t only has a first-order component, $\varepsilon_{t+1}^D+\lambda\frac{\tau_t^D}{\tau}$, while the private signals have zero- and first-order components, the later being identical across investors:

$$\begin{aligned}v_{j,t}^{H,H}(0) &= \epsilon_{j,t}^{H,H} \quad ; \quad v_{j,t}^{H,F}(0)=\epsilon_{j,t}^{H,F} \quad ; \quad v_{j,t}^{H,H}(1)=\varepsilon_{H,t+1} \quad ; \quad v_{j,t}^{H,F}(1)=\varepsilon_{F,t+1}\\ v_{j,t}^{F,H}(0) &= \epsilon_{j,t}^{F,H} \quad ; \quad v_{j,t}^{F,F}(0)=\epsilon_{j,t}^{F,F} \quad ; \quad v_{j,t}^{F,H}(1)=\varepsilon_{H,t+1} \quad ; \quad v_{j,t}^{F,F}(1)=\varepsilon_{F,t+1}\end{aligned}$$

6.5 Expected innovations

Using our results on the orders of coefficients and signals, we compute the various orders of the expected innovations across agents. The zero-order terms are all zero:

$$\left[E_t^{j,H}\varepsilon_{H,t+1}\right](0)=\left[E_t^{j,H}\varepsilon_{F,t+1}\right](0)=\left[E_t^{j,F}\varepsilon_{H,t+1}\right](0)=\left[E_t^{j,F}\varepsilon_{F,t+1}\right](0)=0$$

Agents all have the same first-order expectations:

$$\begin{aligned}\left[E_t^{j,H}\varepsilon_{H,t+1}\right](1) &= \left[E_t^{j,F}\varepsilon_{H,t+1}\right](1) = \frac{1}{2\left(1+2\lambda^2\theta\right)}h_t \\ \left[E_t^{j,H}\varepsilon_{F,t+1}\right](1) &= \left[E_t^{j,F}\varepsilon_{F,t+1}\right](1) = -\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\end{aligned}$$

The expected average and cross-country difference of innovations are then:

$$\begin{aligned}\left[E_t^{j,H}\varepsilon_{t+1}^{A}\right](1) &= \left[E_t^{j,F}\varepsilon_{t+1}^{A}\right](1) = 0 \\ \left[E_t^{j,H}\varepsilon_{t+1}^{D}\right](1) &= \left[E_t^{j,F}\varepsilon_{t+1}^{D}\right](1) = \frac{1}{1+2\lambda^2\theta}h_t\end{aligned}$$

The second-order expectations differ only through the private signals:

$$\begin{aligned}\left[E_t^{j,H}\varepsilon_{H,t+1}\right](2) &= \left[\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}+\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\ \left[E_t^{j,H}\varepsilon_{F,t+1}\right](2) &= \left[\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}+\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\ \left[E_t^{j,F}\varepsilon_{H,t+1}\right](2) &= \left[\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}+\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\ \left[E_t^{j,F}\varepsilon_{F,t+1}\right](2) &= \left[\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}+\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\end{aligned}$$

As private signals average to zero across investors in a given country, all country-wide averages of second-order components are zero.

Finally, the third-order expectations are common to all investors in a given country, but differ across countries:

$$\begin{aligned}\left[E_t^{j,H}\varepsilon_{H,t+1}\right](3) &= \left[\frac{\sigma_{H,H}^2-\left(1+4\lambda^2\theta\right)\sigma_{H,F}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}h_t+\left(1+4\lambda^2\theta\right)\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}+\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\ \left[E_t^{j,H}\varepsilon_{F,t+1}\right](3) &= \left[\frac{\left(1+4\lambda^2\theta\right)\sigma_{H,H}^2-\sigma_{H,F}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}h_t+\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}+\left(1+4\lambda^2\theta\right)\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\ \left[E_t^{j,F}\varepsilon_{H,t+1}\right](3) &= \left[-\frac{\left(1+4\lambda^2\theta\right)\sigma_{H,H}^2-\sigma_{H,F}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}h_t+\left(1+4\lambda^2\theta\right)\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}+\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)} \\ \left[E_t^{j,F}\varepsilon_{F,t+1}\right](3) &= \left[-\frac{\sigma_{H,H}^2-\left(1+4\lambda^2\theta\right)\sigma_{H,F}^2}{2\left(1+2\lambda^2\theta\right)\sigma_{H,H}^2\sigma_{H,F}^2}h_t+\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}+\left(1+4\lambda^2\theta\right)\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}\end{aligned}$$

The expected average and cross-country difference of innovations are then:

$$\begin{aligned}
\left[E_t^{j,H}\varepsilon_{t+1}^A\right](3) &= \left[\left(\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}+\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right)+\frac{\sigma_{H,H}^2-\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]\frac{\sigma_a^2}{2} \\
\left[E_t^{j,F}\varepsilon_{t+1}^A\right](3) &= \left[\left(\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}+\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right)-\frac{\sigma_{H,H}^2-\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]\frac{\sigma_a^2}{2} \\
\left[E_t^{j,H}\varepsilon_{t+1}^D\right](3) &= \left[\left(\frac{\varepsilon_{H,t+1}}{\sigma_{H,H}^2}-\frac{\varepsilon_{F,t+1}}{\sigma_{H,F}^2}\right)-\frac{\sigma_{H,H}^2+\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2 \\
\left[E_t^{j,F}\varepsilon_{t+1}^D\right](3) &= \left[\left(\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2}-\frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right)-\frac{\sigma_{H,H}^2+\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2
\end{aligned}$$

6.6 Expected cross-product of innovations

To compute the expectation fo cross-products of innovations, recall that:

$$E\left(x^2\right)=(Ex)^2+var\left(x\right) \quad ; \quad E\left(xy\right)=cov\left(x,y\right)+\left(Ex\right)\left(Ey\right)$$

We therefore write:

$$\begin{aligned}
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\right] &= \left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)\right]^2+\left[V_t\left(\xi_{t+1}\right)_{1,1}\right] \\
\left[E_t^{j,H}\left(\varepsilon_{F,t+1}\right)^2\right] &= \left[E_t^{j,H}\left(\varepsilon_{F,t+1}\right)\right]^2+\left[V_t\left(\xi_{t+1}\right)_{2,2}\right] \\
\left[E_t^{j,H}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right] &= \left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)\right]\left[E_t^{j,H}\left(\varepsilon_{F,t+1}\right)\right]+\left[V_t\left(\xi_{t+1}\right)_{1,2}\right]
\end{aligned}$$

We focus on the second- and third-order terms of these cross-product.

The second-order terms involve squares of first-order terms and the second-order term of the variance:

$$\begin{aligned}
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\right](2) &= \left[\left[E_t^{j,H}\varepsilon_{H,t+1}\right](1)\right]^2+\left[V_t\left(\xi_{t+1}\right)_{1,1}\right](2) \\
&= \frac{1}{4\left(1+2\lambda^2\theta\right)^2}\left(h_t\right)^2+\frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
\left[E_t^{j,H}\left(\varepsilon_{F,t+1}\right)^2\right](2) &= \left[\left[E_t^{j,H}\varepsilon_{F,t+1}\right](1)\right]^2+\left[V_t\left(\xi_{t+1}\right)_{2,2}\right](2) \\
&= \left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\right](2) \\
\left[E_t^{j,H}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right](2) &= \left[E_t^{j,H}\varepsilon_{H,t+1}\right](1)\left[E_t^{j,H}\varepsilon_{F,t+1}\right](1)+\left[V_t\left(\xi_{t+1}\right)_{1,2}\right](2) \\
&= -\frac{1}{4\left(1+2\lambda^2\theta\right)^2}\left(h_t\right)^2+\frac{1}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2
\end{aligned}$$

As the first-order expected innovations and the second-order variance are the same for all investors, the second-order components of cross-products are the same for all investors. The expectations of cross-products of average and cross-country difference in innovations are:

$$\begin{aligned}
\left[E_t^{j,H}\left(\varepsilon_{t+1}^A\right)^2\right](2) &= \frac{1}{2}\sigma_a^2 \\
\left[E_t^{j,H}\left(\varepsilon_{t+1}^D\right)^2\right](2) &= \frac{1}{\left(1+2\lambda^2\theta\right)^2}\left(h_t\right)^2 + \frac{4\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2 \\
\left[E_t^{j,H}\varepsilon_{t+1}^A\varepsilon_{t+1}^D\right](2) &= 0
\end{aligned}$$

Turning to the third-order component of cross-products, we write for a Home investor (recall that $V_t\left(\xi_{t+1}\right)$ has no third-order terms):

$$\begin{aligned}
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\right](3) &= 2\left[E_t^{j,H}\varepsilon_{H,t+1}\right](1)\left[E_t^{j,H}\varepsilon_{H,t+1}\right](2) \\
&= \left[\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2} + \frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,H}\left(\varepsilon_{F,t+1}\right)^2\right](3) &= 2\left[E_t^{j,H}\varepsilon_{F,t+1}\right](1)\left[E_t^{j,H}\varepsilon_{F,t+1}\right](2) \\
&= -\left[\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2} + \left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,H}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right](3) &= \left[E_t^{j,H}\varepsilon_{H,t+1}\right](1)\left[E_t^{j,H}\varepsilon_{F,t+1}\right](2) \\
&\quad + \left[E_t^{j,H}\varepsilon_{H,t+1}\right](2)\left[E_t^{j,H}\varepsilon_{F,t+1}\right](1) \\
&= \left[\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2} - \frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}\right]\frac{\lambda^2\theta\sigma_a^2}{\left(1+2\lambda^2\theta\right)^2}h_t
\end{aligned}$$

Similarly for a Foreign investor:

$$\begin{aligned}
\left[E_t^{j,F}\left(\varepsilon_{H,t+1}\right)^2\right](3) &= 2\left[E_t^{j,F}\varepsilon_{H,t+1}\right](1)\left[E_t^{j,F}\varepsilon_{H,t+1}\right](2) \\
&= \left[\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}+\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,F}\left(\varepsilon_{F,t+1}\right)^2\right](3) &= 2\left[E_t^{j,F}\varepsilon_{F,t+1}\right](1)\left[E_t^{j,F}\varepsilon_{F,t+1}\right](2) \\
&= -\left[\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}+\left(1+4\lambda^2\theta\right)\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right]\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)^2}h_t \\
\left[E_t^{j,F}\varepsilon_{H,t+1}\varepsilon_{F,t+1}\right](3) &= \left[E_t^{j,F}\varepsilon_{H,t+1}\right](1)\left[E_t^{j,F}\varepsilon_{F,t+1}\right](2) \\
&\quad+\left[E_t^{j,F}\varepsilon_{H,t+1}\right](2)\left[E_t^{j,F}\varepsilon_{F,t+1}\right](1) \\
&= \left[\frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}-\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2}\right]\frac{\lambda^2\theta\sigma_a^2}{\left(1+2\lambda^2\theta\right)^2}h_t
\end{aligned}$$

As the idiosyincratic component of signals averages to zero at the country level, all third-order components of the expectations of cross-products of innovations are zero when expressed as country averages.

6.7 Expected cubic-product of innovations

Turning to the expectations of cubic products, recall that:

$$\begin{aligned}
E\left(x^3\right) &= (Ex)^3+3\,(Ex)\,var\,(x) \\
E\left(yx^2\right) &= 2cov\,(x,y)\,Ex+EyE\left(x^2\right)
\end{aligned}$$

We therefore write the following expectations which are the same for all investors:

$$
\begin{aligned}
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^3\right](3) &= \left[\left[E_t^{j,H}\varepsilon_{H,t+1}\right](1)\right]^3 + 3\left[E_t^{j,H}\varepsilon_{H,t+1}\right](1)\left[V_t\left(\xi_{t+1}\right)_{1,1}\right](2) \\
&= \left[\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]^3 + 3\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\frac{1+4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
\left[E_t^{j,H}\left(\varepsilon_{F,t+1}\right)^3\right](3) &= \left[\left[E_t^{j,H}\varepsilon_{F,t+1}\right](1)\right]^3 + 3\left[E_t^{j,H}\varepsilon_{F,t+1}\right](1)\left[V_t\left(\xi_{t+1}\right)_{2,2}\right](2) \\
&= -\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^3\right](3) \\
\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3) &= 2\left[V_t\left(\xi_{t+1}\right)_{1,2}\right](2)\left[E_t^{j,H}\varepsilon_{H,t+1}\right](1) \\
&\quad + \left[E_t^{j,H}\varepsilon_{F,t+1}\right](1)\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\right](2) \\
&= -\left[\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]^3 + \frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\frac{1-4\lambda^2\theta}{2\left(1+2\lambda^2\theta\right)}\sigma_a^2 \\
\left[E_t^{j,H}\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2\right](3) &= 2\left[V_t\left(\xi_{t+1}\right)_{1,2}\right](2)\left[E_t^{j,H}\varepsilon_{F,t+1}\right](1) \\
&\quad + \left[E_t^{j,H}\varepsilon_{H,t+1}\right](1)\left[\left[E_t^{j,H}\varepsilon_{F,t+1}\right](1)\right]^2 \\
&= -\left[E_t^{j,H}\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3)
\end{aligned}
$$

7 Expected Returns

7.1 Linear terms

It will be useful to compute the various order components of the average expectation of the excess return in (35). We consider the first, second and third order components. These expectations enter the portfolio Euler equations. We compute the orders for the average expectations across Home and Foreign investors, the average across all investors worldwide, and the cross-country difference between Home and Foreign investors. The analysis is undertaken taking account of the fact that the coefficients in (35) themselves have different orders.

Forwarding (38) by one period, h_{t+1} only has first order terms, namely ε_{t+2}^D and τ_{t+1}^D/τ. Notice that at time t agents have no information on these terms, except for the unconditional distribution. In addition, shocks are iid. Therefore:

$$\begin{aligned} E_t h_{t+1} &= 0 \\ E_t \varepsilon^D_{t+1} h_{t+1} &= E_t \varepsilon^D_{t+1} E_t h_{t+1} = 0 \\ E_t \varepsilon^A_{t+1} h_{t+1} &= E_t \varepsilon^A_{t+1} E_t h_{t+1} = 0 \\ E_t \left(h_{t+1}\right)^2 &= Var_t \left(h_{t+1}\right) = 2\sigma_a^2 + 4\lambda^2\theta\sigma_a^2 = 2\left(1+2\lambda^2\theta\right)\sigma_a^2 \end{aligned}$$

The first-order component of the average expectation of (35) is:

$$\begin{aligned} \left[\bar{E}_t^H er_{t+1}\right](1) &= \left[\bar{E}_t^F er_{t+1}\right](1) = \left[\bar{E}_t^A er_{t+1}\right](1) \\ &= lin_t(1) + \mu_{er,1}(0) \frac{1}{1+2\lambda^2\theta} h_t \end{aligned}$$

The second-order component of the average expectation of (35) is:

$$\begin{aligned} \left[\bar{E}_t^H er_{t+1}\right](2) &= \left[\bar{E}_t^F er_{t+1}\right](2) = \left[\bar{E}_t^A er_{t+1}\right](2) \qquad (43) \\ &= lin_t(2) + quad_t(2) + \mu_{er,1}(1) \frac{1}{1+2\lambda^2\theta} h_t + \mu_{er,4}(0) \frac{1}{\left(1+2\lambda^2\theta\right)^2} \left(h_t\right)^2 \\ &\quad + \left[\mu_{er,4}(0) \frac{4\lambda^2\theta}{1+2\lambda^2\theta} + \mu_{er,5}(0) \frac{1}{2} + (1-\delta_1)\mu_{qD}(0)\, 2\left(1+2\lambda^2\theta\right)\right]\sigma_a^2 \end{aligned}$$

where we used that expected cross-products of innovations are the same for all investors.

The third-order component of the expectation of (35) is:

$$\begin{aligned} \left[\bar{E}_t^H er_{t+1}\right](3) &= lin_t(3) + quad_t(3) \\ &\quad + \mu_{er,1}(0) \left[\left(\frac{\varepsilon_{H,t+1}}{\sigma^2_{H,H}} - \frac{\varepsilon_{F,t+1}}{\sigma^2_{H,F}}\right) - \frac{\sigma^2_{H,H} + \sigma^2_{H,F}}{\sigma^2_{H,H}\sigma^2_{H,F}} \frac{1}{2\left(1+2\lambda^2\theta\right)} h_t\right] \frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2 \\ &\quad + \mu_{er,2}(0) \left[\left(\frac{\varepsilon_{H,t+1}}{\sigma^2_{H,H}} + \frac{\varepsilon_{F,t+1}}{\sigma^2_{H,F}}\right) + \frac{\sigma^2_{H,H} - \sigma^2_{H,F}}{\sigma^2_{H,H}\sigma^2_{H,F}} \frac{1}{2\left(1+2\lambda^2\theta\right)} h_t\right] \frac{\sigma_a^2}{2} \\ &\quad + \mu_{er,1}(2) \frac{1}{1+2\lambda^2\theta} h_t + (1-\delta_1)\mu_{qD}(1) 2(1+2\lambda^2\theta)\sigma_a^2 \\ &\quad + \mu_{er,4}(1) \frac{1}{\left(1+2\lambda^2\theta\right)^2}\left(h_t\right)^2 + \left[\mu_{er,4}(1)\frac{4\lambda^2\theta}{1+2\lambda^2\theta} + \mu_{er,5}(1)\frac{1}{2}\right]\sigma_a^2 \\ &\quad + [\bar{E}_t^H cubic_{er}(.)](3) \end{aligned}$$

The corresponding expression for the Foreign country is:

$$\begin{aligned}
\left[\bar{E}_t^F er_{t+1}\right](3) &= lin_t(3) + quad_t(3) \\
&+\mu_{er,1}(0)\left[\left(\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2} - \frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right) - \frac{\sigma_{H,H}^2+\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2 \\
&+\mu_{er,2}(0)\left[\left(\frac{\varepsilon_{H,t+1}}{\sigma_{H,F}^2} + \frac{\varepsilon_{F,t+1}}{\sigma_{H,H}^2}\right) - \frac{\sigma_{H,H}^2-\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{1}{2\left(1+2\lambda^2\theta\right)}h_t\right]\frac{\sigma_a^2}{2} \\
&+\mu_{er,1}(2)\frac{1}{1+2\lambda^2\theta}h_t + (1-\delta_1)\mu_{qD}(1)2(1+2\lambda^2\theta)\sigma_a^2 \\
&+\mu_{er,4}(1)\frac{1}{\left(1+2\lambda^2\theta\right)^2}(h_t)^2 + \left[\mu_{er,4}(1)\frac{4\lambda^2\theta}{1+2\lambda^2\theta} + \mu_{er,5}(1)\frac{1}{2}\right]\sigma_a^2 \\
&+[\bar{E}_t^F cubic_{er}(.)](3)
\end{aligned}$$

The average across the Home and Foreign countries is:

$$\begin{aligned}
\left[\bar{E}_t^A er_{t+1}\right](3) &= lin_t(3) + quad_t(3) \\
&+\mu_{er,1}(0)\frac{\sigma_{H,H}^2+\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\left[\varepsilon_{t+1}^D - \frac{1}{1+2\lambda^2\theta}h_t\right]\frac{\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2 \\
&+\mu_{er,2}(0)\frac{\sigma_{H,H}^2+\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\frac{\sigma_a^2}{2}\varepsilon_{t+1}^A \\
&+\mu_{er,1}(2)\frac{1}{1+2\lambda^2\theta}h_t + (1-\delta_1)\mu_{qD}(1)2(1+2\lambda^2\theta)\sigma_a^2 \\
&+\mu_{er,4}(1)\frac{1}{\left(1+2\lambda^2\theta\right)^2}(h_t)^2 + \left[\mu_{er,4}(1)\frac{4\lambda^2\theta}{1+2\lambda^2\theta} + \mu_{er,5}(1)\frac{1}{2}\right]\sigma_a^2 \\
&+[\bar{E}_t^A cubic_{er}(.)](3)
\end{aligned}$$

The difference across countries is:

$$\begin{aligned}
&\left[\bar{E}_t^H er_{t+1}\right](3) - \left[E_t^F er_{t+1}\right](3) \\
= \ & -\frac{\sigma_{H,H}^2-\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}2\mu_{er,1}(0)\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\varepsilon_{t+1}^A \\
&-\frac{\sigma_{H,H}^2-\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\mu_{er,2}(0)\left[\varepsilon_{t+1}^D - \frac{1}{1+2\lambda^2\theta}h_t\right]\frac{\sigma_a^2}{2} \\
&+[\bar{E}_t^H cubic_{er}(.)](3) - [\bar{E}_t^F cubic_{er}(.)](3)
\end{aligned}$$

The last term, $[\bar{E}_t^H cubic_{er}(.)](3) - [\bar{E}_t^F cubic_{er}(.)](3)$, is equal to zero. This requires some explanation. It involves expectations of cubic products that we do not actually spell out. These products entail variables known at time t, about which investors obviously agree, and innovations. There are three types of terms involving innovations: cubic products of innovations (such as $(\varepsilon_{H,t+1})^2 \varepsilon_{F,t+1}$), quadratic products of innovations times variables known at time t (such as $\varepsilon_{H,t+1}\varepsilon_{F,t+1}S_t$), and quadratic products of variables known at time t times innovations (such as $\varepsilon_{H,t+1}S_t'S_t$). We argue that the investors agree on all three terms.

The third-order component of the expected cubic products of innovations are the same for all investors (see Section 6.7). Next consider the quadratic products of innovations times variables known at time t. The third-order component of these terms depend on the second-order component of the expected cross-products of innovations. These are also the same for all investors (see section 6.6). Finally, consider quadratic products of variables known at time t times innovations. Their third-order components entail the first-order component of the expected innovations. As shown in section 6.5, these are also the same for all investors.

We therefore conclude that $[\bar{E}_t^H cubic_{er}(.)](3) - [\bar{E}_t^F cubic_{er}(.)](3) = 0$, so that

$$\begin{aligned} & \left[\bar{E}_t^H er_{t+1}\right](3) - \left[\bar{E}_t^F er_{t+1}\right](3) \\ = & -\frac{\sigma_{H,H}^2 - \sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2} 2\mu_{er,1}(0)\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\varepsilon_{t+1}^A \\ & -\frac{\sigma_{H,H}^2 - \sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\mu_{er,2}(0)\left[\varepsilon_{t+1}^D - \frac{1}{1+2\lambda^2\theta}h_t\right]\frac{\sigma_a^2}{2} \end{aligned} \tag{44}$$

7.2 Quadratic and cubic terms

We now compute the various components of the terms in (36). The second-order terms are the same for all investors:

$$\begin{aligned} \left[E_t er_{t+1}^2\right](2) = & \left(lin_t(1)\right)^2 + 2\mu_{er,1}(0)\frac{1}{1+2\lambda^2\theta}lin_t(1)h_t + \left(\mu_{er,1}(0)\right)^2\frac{1}{\left(1+2\lambda^2\theta\right)^2}(h_t)^2 \\ & + \left[\left(\mu_{er,1}(0)\right)^2\frac{4\lambda^2\theta}{1+2\lambda^2\theta} + \left(\mu_{er,2}(0)\right)^2\frac{1}{2} + \left(\mu_{er,3}(0)\right)^2 2\left(1+2\lambda^2\theta\right)\right]\sigma_a^2 \end{aligned} \tag{45}$$

The third-order component of $E_t er_{t+1}^2$ is also the same for all investors. It is a bit easier to compute after deriving some of the results from imposing the equations at various orders. We derive an expressions for $[E_t er_{t+1}^2](3)$ in Section 8.3.3. $[E_t er_{t+1}^3](3)$ is also the same across all investors and is equal to zero. We show this in section 8.4.1. It follows directly from the fact that the expected excess return is zero to the first-order, which follows from imposing the portfolio Euler equation to the first-order for individual investors.

8 Imposing orders

8.1 General approach

We now turn to the last overall step towards the solution, which involves imposing the equations to various orders. This will deliver a solution of the control variables as a function of the state variables to various orders.

Following the method developed by Devereux and Sutherland (2010) and Tille and van Wincoop (2010), we do so in two steps. We first combine the second-order component of the difference in portfolio Euler equations (eqn. (28)) and the first-order components of all "other" equations. The "other" equations are the average and the difference of the capital accumulation equations ((18) and (19)), the average and the difference of the market clearing equations ((20) and (21)) and the average portfolio Euler equation (27). Doing so delivers $q_t^D(1)$, $q_t^A(1)$, $k_{t+1}^D(1)$, $k_{t+1}^A(1)$, $z_t^A(1)$ and $z^D(0)$ as a function of the state variables. We refer to the first 5 of these variables, for which we solve the first-order components, as the "other" variables.

Step 2 is the same but one order higher. We impose the third-order component of the difference in portfolio Euler equations (eqn.(28)) and the second-order components of all other equations. Doing so delivers $q_t^D(2)$, $q_t^A(2)$, $k_{t+1}^D(2)$, $k_{t+1}^A(2)$, $z_t^A(2)$ and $z^D(1)$ as a function of the state variables.

One final step is needed that does not arise in models without information heterogeneity. The solution is conditional on the signal to noise ratio λ in (38). We solve for λ by equating the first-order component of equity demand to the first-order component of equity supply. The former, $z_t^A(1)$ from a demand perspective, follows from imposing the third-order component of the average portfolio Euler equation (27), while the latter follows from the first-order component of the average market clearing condition (20).

Each of the three steps is detailed in this section.

8.2 First Step: First-Order Solution

We start with the first-order component of the "other" variables and the zero-order component of the cross-country difference in portfolio shares.

8.2.1 "Other" equations

We start with the worldwide average of the "other" variables and equations. The first-order components of the average capital accumulation (18) and the average asset market clearing equation (20) imply:

$$q_t^A(1) = \frac{\xi}{1+\xi} a_t^A(1) - \frac{\omega\xi}{1+\xi} k_t^A(1)$$

which immediately gives the zero-order coefficents in (31):

$$\alpha_{qA}(0) = \frac{\xi}{1+\xi}\begin{bmatrix} 0 & 1 & 0 & -\omega \end{bmatrix} = \frac{\xi}{1+\xi}(I_2 - \omega I_4) \tag{46}$$

and $\alpha_{5,qA}(0) = 0$. The average capital accumulation equation (18) in turns implies:

$$k_{t+1}^A(1) = \alpha_{kA}(0) S_t = \left[\frac{1}{1+\xi} I_2 + \frac{1+\xi-\omega}{1+\xi} I_4\right] S_t(1) \tag{47}$$

The first-order components of the difference in capital accumulation (19) and the difference in asset market clearing equations (21) give the solution for $k_{t+1}^D(1)$ and $z_t^A(1)$ as functions of $q_t^D(1)$:

$$\begin{aligned} k_{t+1}^D(1) &= \left[\frac{1}{\xi}\alpha_{qD}(0) + I_3\right] S_t(1) + \frac{1}{\xi}\alpha_{5,qD}(0) h_t && (48) \\ z_t^A(1) &= \frac{1}{4}\left[\frac{1+\xi}{\xi}\alpha_{qD}(0) + I_3 - z^D(0)(I_1 + (1-\omega) I_3)\right] S_t(1) \\ &\quad + \frac{1+\xi}{4\xi}\alpha_{5,qD}(0) h_t && (49) \end{aligned}$$

We finally need to solve for $\alpha_{qD}(0)$ and $\alpha_{5,qD}(0)$ in the first-order component of the difference in asset prices: $q_t^D(1) = \alpha_{qD}(0)S_t(1) + \alpha_{5,qD}(0)h_t$. We do so by imposing the the first-order component of the average portfolio Euler equation (27). This implies that the first-order expected excess return, which is the same for all agents (as shown in section 7.1), is zero:

$$0 = \left[\bar{E}_t^A er_{t+1}\right](1) = lin_t(1) + \mu_{er,1}(0)\frac{1}{1+2\lambda^2\theta} h_t \tag{50}$$

where, using our results from section 5.3:

$$\begin{aligned}
\mu_{er,1}(0) &= (1-\delta_1)\alpha_{1,qD}(0)+\delta_1 \\
lin_t(1) &= \left[(1-\delta_1)\alpha_{qD}(0)+\delta_1(I_1-\omega I_3)\right]\bar{S}_{t+1}(1)-\alpha_{qD}(0)\,S_t(1)-\alpha_{5,qD}(0)\,h_t
\end{aligned}$$

The first-order component of $\bar{S}_{t+1}$ in (32) are:

$$\bar{S}_{t+1}(1) = N_1(0)\,S_t(1)+N_3(0)\,h_t$$

where:

$$\begin{aligned}
N_1(0) &= \begin{vmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ \frac{1}{\xi}\alpha_{1,qD}(0) & \frac{1}{\xi}\alpha_{2,qD}(0) & \frac{1}{\xi}\alpha_{3,qD}(0)+1 & \frac{1}{\xi}\alpha_{4,qD}(0) \\ 0 & \frac{1}{1+\xi} & 0 & \frac{1+\xi-\omega}{1+\xi} \end{vmatrix} \\
N_3(0) &= \begin{vmatrix} 0 \\ 0 \\ \frac{1}{\xi}\alpha_{5,qD}(0) \\ 0 \end{vmatrix}
\end{aligned}$$

We then write:

$$\begin{aligned}
lin_t(1) &= \left[\left[(1-\delta_1)\alpha_{qD}(0)+\delta_1(I_1-\omega I_3)\right]N_1(0)-\alpha_{qD}(0)\right]S_t(1) \\
&\quad+\left[\frac{(1-\delta_1)}{\xi}\alpha_{3,qD}(0)-\frac{\omega\delta_1}{\xi}-1\right]\alpha_{5,qD}(0)\,h_t \\
&= \begin{vmatrix} \alpha_{1,qD}(0)\left[\left[(1-\delta_1)\rho-1\right]-\frac{\omega}{\xi}\delta_1+\alpha_{3,qD}(0)\,\frac{1-\delta_1}{\xi}\right]+\delta_1\rho \\ \alpha_{2,qD}(0)\left[\left[(1-\delta_1)\rho-1\right]+\alpha_{3,qD}(0)\,\frac{1-\delta_1}{\xi}-\delta_1\frac{\omega}{\xi}\right]+\frac{1-\delta_1}{1+\xi}\alpha_{4,qD}(0) \\ \alpha_{3,qD}(0)\left[\frac{1-\delta_1}{\xi}\alpha_{3,qD}(0)-\delta_1\frac{\xi+\omega}{\xi}\right]-\omega\delta_1 \\ \left[\alpha_{3,qD}(0)\,\frac{1-\delta_1}{\xi}+(1-\delta_1)\frac{1+\xi-\omega}{1+\xi}-1-\delta_1\frac{\omega}{\xi}\right]\alpha_{4,qD}(0) \end{vmatrix}' S_t(1) \\
&\quad+\left[\frac{(1-\delta_1)}{\xi}\alpha_{3,qD}(0)-\frac{\omega\delta_1}{\xi}-1\right]\alpha_{5,qD}(0)\,h_t
\end{aligned}$$

We now use the method of undetermined coefficients in (50). The coefficients on $k_t^A(1)$ imply $\alpha_{4,qD}(0)=0$, and the coefficient on $a_t^A(1)$ in turns implies $\alpha_{2,qD}(0)=0$.

From (48) the coefficient on $k_t^D(1)$ in the solution for $k_{t+1}^D(1)$ is equal to $\alpha_{3,kD}(0)=\alpha_{3,qD}(0)/\xi+1$, so that $\alpha_{3,qD}(0)=\xi\left[\alpha_{3,kD}(0)-1\right]$. Using this, and setting the coefficient on $k_t^D(1)$ in (50) equal to zero, gives the following quadratic polynomial in $\alpha_{3,kD}(0)$:

$$\begin{aligned} 0 &= Pol\left(\alpha_{3,kD}(0)\right) \\ &= (1-\delta_1)\,\xi\left[\alpha_{3,kD}(0)-1\right]^2 - \delta_1(\xi+\omega)\left[\alpha_{3,kD}(0)-1\right] - \omega\delta_1 \end{aligned}$$

The polynomial is positive for $\alpha_{3,kD}(0)$ being minus or plus infinity, positive for $\alpha_{3,kD}(0)=0$, and negative for $\alpha_{3,kD}(0)=1$. There is thus an explosive root above one, and a stable one between zero and one which we consider. This implies $\alpha_{3,qD}(0)<0$. Specifically:

$$\alpha_{3,qD}(0) = \frac{\delta_1(\xi+\omega) - \left[(\delta_1)^2(\xi+\omega)^2 + 4\delta_1(1-\delta_1)\xi\omega\right]^{0.5}}{2(1-\delta_1)} \tag{51}$$

The coefficient on $a_t^D(1)$ implies:

$$\alpha_{1,qD}(0) = \frac{\delta_1\rho}{1-(1-\delta_1)\rho + \left[\omega\delta_1 - (1-\delta_1)\alpha_{3,qD}(0)\right]\frac{1}{\xi}} > 0 \tag{52}$$

The coefficient on h_t gives the solution for $\alpha_{5,qD}(0)$, which is conditional on the noise to signal ratio λ that we have yet to solve for:

$$\alpha_{5,qD}(0) = \frac{(1-\delta_1)\alpha_{1,qD}(0)+\delta_1}{1+\left[\omega\delta_1 - (1-\delta_1)\alpha_{3,qD}(0)\right]\frac{1}{\xi}}\frac{1}{1+2\lambda^2\theta} > 0 \tag{53}$$

This completes the solution of $q_t^D(1)$, $q_t^A(1)$, $k_{t+1}^D(1)$, $k_{t+1}^A(1)$, $z_t^A(1)$ as a function of $S_t(1)$ and h_t. Note that the solution for $\alpha_{5,qD}(0)$ is conditional on λ, which will be solved later. Also, the solution for $z_t^A(1)$ is conditional on $z^D(0)$, which we now solve.

8.2.2 Difference in portfolio Euler condition

We solve for $z^D(0)$ by imposing the second-order component of the difference in portfolio Euler equations (28), using the results from the first-order solution above.

The second order component of (28) is written as:

$$z^D(0)\left[E_t er_{t+1}^2\right](2) = \left[\bar{E}_t^H er_{t+1}\right](2) - \left[\bar{E}_t^F er_{t+1}\right](2) + 2\tau$$

(43) shows that the second-order component of the expected excess return is the same for all agents, so we get: $\left[\bar{E}_t^H er_{t+1}\right](2) - \left[\bar{E}_t^F er_{t+1}\right](2) = 0$. Therefore

$$z^D(0) = \frac{2\tau}{\left[E_t er_{t+1}^2\right](2)}$$

$\left[E_t er_{t+1}^2\right](2)$ is taken from (45). Using our results from section 5.3 and 8.2.1, we write:

$$\begin{aligned}\mu_{er,1}(0) &= (1-\delta_1)\alpha_{1,qD}(0)+\delta_1\\ \mu_{er,2}(0) &= (1-\delta_1)\alpha_{2,qD}(0)=0\\ \mu_{er,3}(0) &= (1-\delta_1)\alpha_{5,qD}(0)\end{aligned}$$

Using these results, along with $lin_t(1) = -\mu_{er,1}(0)\left(1+2\lambda^2\theta\right)^{-1} h_t$ from (50), (45) then becomes:

$$\begin{aligned}&\left[E_t er_{t+1}^2\right](2)\\ =& \left[\left(\mu_{er,1}(0)\right)^2 \frac{4\lambda^2\theta}{1+2\lambda^2\theta} + \left(\mu_{er,2}(0)\right)^2 \frac{1}{2} + \left(\mu_{er,3}(0)\right)^2 2\left(1+2\lambda^2\theta\right)\right]\sigma_a^2 \qquad (54)\\ =& \frac{2\left[(1-\delta_1)\alpha_{1,qD}(0)+\delta_1\right]^2}{1+2\lambda^2\theta}\left[2\lambda^2\theta + \left(\frac{(1-\delta_1)}{1+\left[\omega\delta_1-(1-\delta_1)\alpha_{3,qD}(0)\right]\frac{1}{\xi}}\right)^2\right]\sigma_a^2\end{aligned}$$

At this stage we have solved for the first-order components of all "other" variables and the zero-order component of the cross country difference in the portfolio shares, $z^D(0)$. The solution conditional, as $\alpha_{5,qD}(0)$ depends on λ. We next turn to the second step of the solution, which parallels the first step but one order higher.

8.3 Second-Order Solution

We now solve for the second-order component of the "other" variables and the first-order component of the cross-country difference in portfolio shares.

8.3.1 Worldwide Average Variables

We start by computing the worldwide average variables $k_{t+1}^A(2)$, $q_t^A(2)$ and $z_t^A(2)$. These are solved using the second-order component of the average capital accumulation equation (18), the average asset market clearing condition (20) and the difference in asset market clearing conditions (21).

The second-order component of the average capital accumulation equation (18) is:

$$k_{t+1}^A(2) = k_t^A(2) + \frac{1}{\xi} q_t^A(2) + \frac{\xi - 1}{2\xi^2} S_t'(1) N S_t(1) + \frac{\xi - 1}{4\xi^2} \alpha_{5,qD}(0) \alpha_{qD}(0) S_t(1) h_t + \frac{\xi - 1}{8\xi^2} (\alpha_{5,qD}(0))^2 (h_t)^2 \tag{55}$$

where:

$$N = \frac{1}{4} \alpha'_{qD}(0) \alpha_{qD}(0) + \alpha'_{qA}(0) \alpha_{qA}(0)$$

The average asset market clearing condition (20) then implies:

$$\begin{aligned} q_t^A(2) &= \frac{\xi}{1+\xi} (I_2 - \omega I_4) S_t(2) \\ &+ \frac{\xi}{1+\xi} \frac{1}{8} S_t'(1) \begin{bmatrix} -[G_{qA,1}]' G_{qA,1} - 4\frac{\xi-1}{\xi^2} N \\ + (I_1 + (1-\omega) I_3)' (I_1 + (1-\omega) I_3) \end{bmatrix} S_t(1) \\ &- \frac{\xi}{1+\xi} \frac{3+\xi}{8\xi} (\alpha_{5,qD}(0))^2 (h_t)^2 \\ &- \frac{\xi}{1+\xi} \frac{1}{4} \left[\frac{3+\xi}{\xi} \alpha_{qD}(0) + I_3 \frac{1+\xi}{\xi} \right] \alpha_{5,qD}(0) S_t(1) h_t \end{aligned} \tag{56}$$

where:

$$G_{qA,1} = \frac{1+\xi}{\xi} \alpha_{qD}(0) + I_3$$

Using the conjecture for q_t^A (31), it then follows that the second-order component of the average asset price is

$$\begin{aligned} q_t^A(2) &= \alpha_{qA}(0) S_t(2) + \alpha_{qA}(1) S_t(1) + \alpha_{5,qA}(1) h_1 \\ &+ S_t(1)' A_{qA}(0) S_t(1) + \beta_{qA}(0) S_t(1) h_t + \mu_{qA}(0) h_t^2 \end{aligned} \tag{57}$$

We solve for the coefficients by combining (56)-(57) and using the method of undetermined coefficients. As shown in the previous section, this implies $\alpha_{qA}(0) = \xi(1+\xi)^{-1}(I_2 - \omega I_4)$. Furthermore, we get:

$$\alpha_{qA}(1) = \alpha_{5,qA}(1) = 0 \tag{58}$$

$$A_{qA}(0) = \frac{\xi}{1+\xi} \frac{1}{8} \begin{bmatrix} -[G_{qA,1}]' G_{qA,1} - 4\frac{\xi-1}{\xi^2} N \\ + (I_1 + (1-\omega) I_3)' (I_1 + (1-\omega) I_3) \end{bmatrix} \tag{59}$$

$$\beta_{qA}(0) = -\frac{\xi}{1+\xi} \frac{1}{4} \left[\frac{3+\xi}{\xi} \alpha_{qD}(0) + I_3 \frac{1+\xi}{\xi} \right] \alpha_{5,qD}(0) \tag{60}$$

$$\mu_{qA}(0) = -\frac{\xi}{1+\xi} \frac{3+\xi}{8\xi} (\alpha_{5,qD}(0))^2 \tag{61}$$

Substituting this into (55) gives the second-order component of the average capital stock at $t+1$:

$$
\begin{aligned}
k_{t+1}^{A}(2) \quad = \quad & \left(\frac{1}{1+\xi} I_2 + \frac{1+\xi-\omega}{1+\xi} I_4\right) S_t(2) \qquad (62)\\
& + S_t'(1)\left[\frac{1}{\xi} A_{qA}(0) + \frac{\xi-1}{2\xi^2} N\right] S_t(1) \\
& + \frac{1}{4}\left[-\frac{1}{\xi^2}\frac{1+3\xi}{1+\xi}\alpha_{qD}(0) - I_3\frac{1}{\xi}\right]\alpha_{5,qD}(0)\, S_t(1)\, h_t \\
& - \frac{1}{8\xi^2}\frac{1+3\xi}{1+\xi}\left(\alpha_{5,qD}(0)\right)^2 (h_t)^2
\end{aligned}
$$

The last world-wide average variable that we need to compute is $z_t^A(2)$. This follows from the second-order component of the difference in asset market clearing conditions (21), which gives:

$$
\begin{aligned}
z_t^A(2) \quad = \quad & \frac{1}{4}\left[q_t^D(2) + k_{t+1}^D(2) - z_t^D(1)\left(I_1 + (1-\omega) I_3\right) S_t(1)\right] \qquad (63)\\
& - \frac{1}{4} z^D(0)\left(I_1 + (1-\omega) I_3\right) S_t(2)
\end{aligned}
$$

This depends on $q_t^D(2)$ and $k_{t+1}^D(2)$, which will be solved in the next subsection, and on $z_t^D(1)$, which is solved in Section 8.3.3.

8.3.2 Cross-Country Difference Variables

We next turn to computing the cross-country differences $q_t^D(2)$ and $k_{t+1}^D(2)$. These are computed using the second-order component of the average portfolio Euler equation (27) and the difference in capital accumulation equations (19).

The second-order component of the average portfolio Euler equation (27), along with $z^A(0) = 0.5$, implies that $\left[\bar{E}_t^A er_{t+1}\right](2) = 0$. Using (43) we write:

$$
\begin{aligned}
0 = {} & lin_t(2) + quad_t(2) + \mu_{er,1}(1)\frac{1}{1+2\lambda^2\theta} h_t + \mu_{er,4}(0)\frac{1}{\left(1+2\lambda^2\theta\right)^2}(h_t)^2 \\
& + \left[\mu_{er,4}(0)\frac{4\lambda^2\theta}{1+2\lambda^2\theta} + \mu_{er,5}(0)\frac{1}{2} + (1-\delta_1)\mu_{qD}(0)\, 2\left(1+2\lambda^2\theta\right)\right]\sigma_a^2 \qquad (64)
\end{aligned}
$$

Using $\alpha_{2,qD}(0) = \alpha_{1,qA}(0) = 0$ and the expressions in section 5.3 we write:

$$\begin{aligned}
\mu_{er,1}(1) &= (1-\delta_1)\,\alpha_{1,qD}(1) + 2(1-\delta_1)A_{qD,\text{row 1}}(0)\,\bar{S}_{t+1}(1) \\
&\quad +\delta_2(1-\alpha_{1,qD}(0))\,(I_2 - \omega I_4 - \alpha_{qA}(0))\,\bar{S}_{t+1}(1) \\
\mu_{er,4}(0) &= (1-\delta_1)A_{11,qD}(0) \\
\mu_{er,5}(0) &= (1-\delta_1)A_{22,qD}(0)
\end{aligned}$$

The term $quad_t(2)$ involves the term

$$\delta_2 \bar{S}'_{t+1}\,(I_1 - \omega I_3 - \alpha_{qD})'\,(I_2 - \omega I_4 - \alpha_{qA})\,\bar{S}_{t+1}$$

We will write this as $(1-\delta_1)\bar{S}'_{t+1} H \bar{S}_{t+1}$ with

$$H = \frac{\delta_2}{2(1-\delta_1)}\,(I_1 - \omega I_3 - \alpha_{qD})'\,(I_2 - \omega I_4 - \alpha_{qA}) + \frac{\delta_2}{2(1-\delta_1)}\,(I_2 - \omega I_4 - \alpha_{qA})'\,(I_1 - \omega I_3 - \alpha_{qD}) \tag{65}$$

Note that it takes the form

$$H = \begin{pmatrix} 0 & H_{12} & 0 & H_{14} \\ H_{12} & 0 & H_{23} & 0 \\ 0 & H_{23} & 0 & H_{34} \\ H_{14} & 0 & H_{34} & 0 \end{pmatrix} \tag{66}$$

We then get:

$$\begin{aligned}
quad_t(2) &= (1-\delta_1)\bar{S}'_{t+1}(1)\,(A_{qD}(0) + H(0))\bar{S}_{t+1}(1) - S'_t(1)\,A_{qD}(0)\,S_t(1) \\
&\quad -\beta_{qD}(0)\,S_t(1)\,h_t - \mu_{qD}(0)\,(h_t)^2
\end{aligned}$$

We also write:

$$\begin{aligned}
lin_t(2) &= \left[(1-\delta_1)\alpha_{qD}(0) + \delta_1(I_1 - \omega I_3)\right]\bar{S}_{t+1}(2) - \alpha_{qD}(0)\,S_t(2) \\
&\quad +\alpha_{qD}(1)\left[(1-\delta_1)\bar{S}_{t+1}(1) - S_t(1)\right] - \alpha_{5,qD}(1)\,h_t
\end{aligned}$$

Using $\bar{S}_{t+1}(1) = N_1(0)\,S_t(1) + N_3(0)\,h_t$ and the fact that only the third and fourth elements of $\bar{S}_{t+1}$ and S_t have non-zero second-order terms, we write:

$$\begin{aligned}
lin_t(2) &= \left[(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1\right]k^D_{t+1}(2) - \alpha_{3,qD}(0)\,k^D_t(2) \\
&\quad +\alpha_{qD}(1)\left[(1-\delta_1)N_1(0) - I_{4x4}\right]S_t(1) \\
&\quad +\left[(1-\delta_1)\alpha_{qD}(1)\,N_3(0) - \alpha_{5,qD}(1)\right]h_t \\
quad_t(2) &= S_t(1)'\left[(1-\delta_1)\left[N_1(0)\right]'(A_{qD}(0) + H(0))\,N_1(0) - A_{qD}(0)\right]S_t(1) \\
&\quad +\left[2(1-\delta_1)N'_3(0)\,(A_{qD}(0) + H(0))\,N_1(0) - \beta_{qD}(0)\right]S_t(1)\,h_t \\
&\quad +\left[(1-\delta_1)N'_3(0)\,(A_{qD}(0) + H(0))\,N_3(0) - \mu_{qD}(0)\right](h_t)^2
\end{aligned}$$

The second-order component of (19), the difference in capital accumulation equations, implies that:

$$
\begin{aligned}
k_{t+1}^{D}(2) &= k_{t}^{D}(2) + \frac{1}{\xi} q_{t}^{D}(2) + \frac{\xi-1}{\xi^{2}} q_{t}^{A}(1)\, q_{t}^{D}(1) \\
&= \left(I_3 + \frac{1}{\xi}\alpha_{qD}(0)\right) S_t(2) + \frac{1}{\xi}\alpha_{qD}(1)\, S_t(1) + \frac{1}{\xi}\alpha_{5,qD}(1)\, h_t + \frac{1}{\xi}\mu_{qD}(0)\,(h_t)^2 \\
&\quad + S_t'(1)\left[\frac{1}{\xi}A_{qD}(0) + 0.5\frac{\xi-1}{\xi^2}\left(\alpha_{qA}(0)\right)'\alpha_{qD}(0) + 0.5\left(\alpha_{qD}(0)\right)'\alpha_{qA}(0)\right] S_t(1) \\
&\quad + \left[\frac{\xi-1}{\xi^2}\alpha_{5,qD}(0)\,\alpha_{qA}(0) + \frac{1}{\xi}\beta_{qD}(0)\right] S_t(1)\, h_t \qquad (67)
\end{aligned}
$$

Combining all elements (64) becomes:

$$
\begin{aligned}
0 = {} & [(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1] \\
& \times \left[\begin{array}{c}
\left(1+\frac{1}{\xi}\alpha_{3,qD}(0)\right) I_3 S_t(2) + \frac{1}{\xi}\alpha_{qD}(1)\, S_t(1) \\
+\frac{1}{\xi}\alpha_{5,qD}(1)\, h_t + \frac{1}{\xi}\mu_{qD}(0)\,(h_t)^2 \\
+S_t'(1)\left[\frac{1}{\xi}A_{qD}(0) + \frac{1}{2}\frac{\xi-1}{\xi^2}\left[(\alpha_{qA}(0))'\alpha_{qD}(0) + (\alpha_{qD}(0))'\alpha_{qA}(0)\right]\right] S_t(1) \\
+\left[\frac{\xi-1}{\xi^2}\alpha_{5,qD}(0)\,\alpha_{qA}(0) + \frac{1}{\xi}\beta_{qD}(0)\right] S_t(1)\, h_t
\end{array}\right] \\
& -\alpha_{3,qD}(0)\, I_3 S_t(2) + \alpha_{qD}(1)\left[(1-\delta_1)N_1(0) - I_{4x4}\right] S_t(1) \\
& + \left[(1-\delta_1)\alpha_{qD}(1)\, N_3(0) - \alpha_{5,qD}(1)\right] h_t \qquad (68) \\
& + S_t(1)'\left[(1-\delta_1)\left[N_1(0)\right]'\left(A_{qD}(0) + H(0)\right) N_1(0) - A_{qD}(0)\right] S_t(1) \\
& + \left[2(1-\delta_1)N_3'(0)\left(A_{qD}(0) + H(0)\right) N_1(0) - \beta_{qD}(0)\right] S_t(1)\, h_t \\
& + \left[(1-\delta_1)N_3'(0)\left(A_{qD}(0) + H(0)\right) N_3(0) - \mu_{qD}(0)\right](h_t)^2 \\
& + (1-\delta_1)\,\alpha_{1,qD}(1)\frac{1}{1+2\lambda^2\theta} h_t \\
& + \left[\begin{array}{c} 2(1-\delta_1)A_{qD,\text{row }1}(0) \\ +\delta_2(1-\alpha_{1,qD}(0))\,(I_2 - \omega I_4) \end{array}\right] \frac{1}{1+2\lambda^2\theta} N_1(0)\, S_t(1)\, h_t \\
& + \left[\begin{array}{c} 2(1-\delta_1)A_{qD,\text{row }1}(0) \\ +\delta_2(1-\alpha_{1,qD}(0))\,(I_2 - \omega I_4) \end{array}\right] N_3(0)\frac{1}{1+2\lambda^2\theta}(h_t)^2 \\
& + (1-\delta_1)A_{11,qD}(0)\frac{1}{\left(1+2\lambda^2\theta\right)^2}(h_t)^2 \\
& + \left[\begin{array}{c} (1-\delta_1)A_{11,qD}(0)\frac{4\lambda^2\theta}{1+2\lambda^2\theta} + (1-\delta_1)A_{22,qD}(0)\frac{1}{2} \\ +(1-\delta_1)\mu_{qD}(0)\, 2\left(1+2\lambda^2\theta\right) \end{array}\right]\sigma_a^2
\end{aligned}
$$

Setting the coeffficient on $S_t(1)$ to zero implies $\alpha_{qD}(1) = 0$. Setting the coeffficient on h_t to zero then implies $\alpha_{5,qD}(1) = 0$. The coefficient on $S_t(2)$ gives $\alpha_{3,qD}(0)$ as before.

The coefficient on $S_t'(1)\,S_t(1)$ in (68) implies:

$$
\begin{aligned}
0 &= \left[[(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1]\frac{1}{\xi} - 1\right]\frac{1}{1-\delta_1}A_{qD}(0) \\
&+ \frac{(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1}{1-\delta_1}\frac{1}{2}\frac{\xi-1}{\xi^2}\left[(\alpha_{qA}(0))'\,\alpha_{qD}(0) + (\alpha_{qD}(0))'\,\alpha_{qA}(0)\right] \\
&+ [N_1(0)]'\,(A_{qD}(0) + H(0))N_1(0)
\end{aligned}
\tag{69}
$$

This gives the solution for the matrix $A_{qD}(0)$. The model is simple enough to solve for it analytically, using the fact that $N_1(0)$ and $(\alpha_{qA}(0))'\,\alpha_{qD}(0)$ contain many zeros:

$$
\begin{aligned}
&\frac{1}{2}\left[(\alpha_{qA}(0))'\,\alpha_{qD}(0) + (\alpha_{qD}(0))'\,\alpha_{qA}(0)\right] \\
= &\ \frac{1}{2}\frac{\xi}{1+\xi}\begin{pmatrix} 0 & \alpha_{1,qD}(0) & 0 & -\omega\alpha_{1,qD}(0) \\ \alpha_{1,qD}(0) & 0 & \alpha_{3,qD}(0) & 0 \\ 0 & \alpha_{3,qD}(0) & 0 & -\omega\alpha_{3,qD}(0) \\ -\omega\alpha_{1,qD}(0) & 0 & -\omega\alpha_{3,qD}(0) & 0 \end{pmatrix}
\end{aligned}
$$

We then get the symmetric matrix $A_{qD}(0)$:

$$
A_{qD}(0) = \begin{pmatrix} 0 & A_{12,qD}(0) & 0 & A_{14,qD}(0) \\ A_{12,qD}(0) & 0 & A_{23,qD}(0) & 0 \\ 0 & A_{23,qD}(0) & 0 & A_{34,qD}(0) \\ A_{14,qD}(0) & 0 & A_{34,qD}(0) & 0 \end{pmatrix}
$$

where $A_{34,qD}(0)$ solves:

$$
\begin{aligned}
0 &= \left[[(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1]\frac{1}{\xi} - 1\right]\frac{1}{1-\delta_1}A_{34,qD}(0) \\
&- \frac{(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1}{1-\delta_1}\frac{\xi}{1+\xi}\frac{\omega}{2}\alpha_{3,qD}(0)\frac{\xi-1}{\xi^2} \\
&+ \frac{1+\xi-\omega}{1+\xi}\left(\frac{1}{\xi}\alpha_{3,qD}(0) + 1\right)(A_{34,qD}(0) + H_{34}(0))
\end{aligned}
$$

and $A_{23,qD}(0)$ is computed from:

$$\begin{aligned}
0 &= \left[[(1-\delta_1)\alpha_{3,qD}(0)-\omega\delta_1]\frac{1}{\xi}-1\right]\frac{1}{1-\delta_1}A_{23,qD}(0) \\
&\quad +\frac{(1-\delta_1)\alpha_{3,qD}(0)-\omega\delta_1}{1-\delta_1}\frac{\xi}{1+\xi}\frac{1}{2}\alpha_{3,qD}(0)\frac{\xi-1}{\xi^2} \\
&\quad +\left(\frac{1}{\xi}\alpha_{3,qD}(0)+1\right)\left[\rho(A_{23,qD}(0)+H_{23}(0))+\frac{1}{1+\xi}(A_{34,qD}(0)+H_{34}(0))\right]
\end{aligned}$$

$A_{14,qD}(0)$ is computed from:

$$\begin{aligned}
0 &= \left[[(1-\delta_1)\alpha_{3,qD}(0)-\omega\delta_1]\frac{1}{\xi}-1\right]\frac{1}{1-\delta_1}A_{14,qD}(0) \\
&\quad -\frac{(1-\delta_1)\alpha_{3,qD}(0)-\omega\delta_1}{1-\delta_1}\frac{\xi}{1+\xi}\frac{\omega}{2}\alpha_{1,qD}(0)\frac{\xi-1}{\xi^2} \\
&\quad +\frac{1+\xi-\omega}{1+\xi}\left[\rho(A_{14,qD}(0)+H_{14}(0))+\frac{1}{\xi}\alpha_{1,qD}(0)(A_{34,qD}(0)+H_{34}(0))\right]
\end{aligned}$$

and $A_{12,qD}(0)$ is computed from:

$$\begin{aligned}
0 &= \left[[(1-\delta_1)\alpha_{3,qD}(0)-\omega\delta_1]\frac{1}{\xi}-1\right]\frac{1}{1-\delta_1}A_{12,qD}(0) \\
&\quad +\frac{(1-\delta_1)\alpha_{3,qD}(0)-\omega\delta_1}{1-\delta_1}\frac{\xi}{1+\xi}\frac{1}{2}\alpha_{1,qD}(0)\frac{\xi-1}{\xi^2} \\
&\quad +\rho\left[\rho(A_{12,qD}(0)+H_{12}(0))+\frac{1}{\xi}\alpha_{1,qD}(0)(A_{23,qD}(0)+H_{23}(0))\right] \\
&\quad +\frac{1}{1+\xi}\left[\rho(A_{14,qD}(0)+H_{14}(0))+\frac{1}{\xi}\alpha_{1,qD}(0)(A_{34,qD}(0)+H_{34}(0))\right]
\end{aligned}$$

Note that this implies:

$$\begin{aligned}
& S_t'(1)\,A_{qD}(0)\,S_t(1) \\
=\ & 2A_{12,qD}(0)\,a_t^A(1)\,a_t^D(1)+2A_{23,qD}(0)\,k_t^D(1)\,a_t^A(1) \\
& +2A_{14,qD}(0)\,k_t^A(1)\,a_t^D(1)+2A_{34,qD}(0)\,k_t^A(1)\,k_t^D(1)
\end{aligned}$$

Having solved for $A_{qD}(0)$,we use our results as well as $N_3(0)$ to write the coefficient on $(h_t)^2$ in (68) as:

$$\begin{aligned}
0 &= \left[[(1-\delta_1)\alpha_{3,qD}(0)-\omega\delta_1]\frac{1}{\xi}-1\right]\mu_{qD}(0) \\
&\quad +2(1-\delta_1)A_{13qD}(0)\frac{1}{\xi}\alpha_{5,qD}(0)\frac{1}{1+2\lambda^2\theta} \\
&\quad +A_{11,qD}(0)\frac{1-\delta_1}{(1+2\lambda^2\theta)^2}
\end{aligned}$$

Since $A_{11,qD}(0) = A_{13,qD}(0) = 0$, we conclude that $\mu_{qD}(0) = 0$.

The coefficient on $S_t(1)\, h_t$ in (68) implies:

$$
\begin{aligned}
& [(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1] \frac{\xi - 1}{\xi^2} \alpha_{5,qD}(0)\, \alpha_{qA}(0) \\
& + \left[[(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1] \frac{1}{\xi} - 1 \right] \beta_{qD}(0) \\
& + 2(1-\delta_1) \left| \begin{array}{c} 0 \\ \rho \left[\frac{1}{\xi}\alpha_{5,qD}(0) \left[A_{23,qD}(0) + H_{23}(0) \right] + \frac{A_{12,qD}(0)}{1+2\lambda^2\theta} \right] + \frac{1}{1+\xi} \left[\frac{1}{\xi}\alpha_{5,qD}(0) \left[A_{34,qD}(0) + H_{34}(0) \right] + \right. \\ 0 \\ \frac{1+\xi-\omega}{1+\xi} \left[\frac{1}{\xi}\alpha_{5,qD}(0) \left[A_{34,qD}(0) + H_{34}(0) \right] + \frac{A_{14,qD}(0)}{1+2\lambda^2\theta} \right] \end{array} \right. \\
& + \frac{\delta_2(1-\alpha_{1,qD}(0))}{1+2\lambda^2\theta} \left| \begin{array}{c} 0 \\ \rho - \omega\frac{1}{1+\xi} \\ 0 \\ -\omega\frac{1+\xi-\omega}{1+\xi} \end{array} \right|' = 0
\end{aligned}
$$

which gives the solution for $\beta_{qD}(0)$. As $\alpha_{qA}(0) = \xi(1+\xi)^{-1}(I_2 - \omega I_4)$, we conclude that $\beta_{1,qD}(0) = \beta_{3,qD}(0) = 0$.

To summarize, using the second-order component of the average portfolio Euler equation (27) and the difference in capital accumulation equation (19) we have now computed $k^D_{t+1}(2)$ in (67) and $q^D_t(2)$. The latter is equal to

$$q^D_t(2) = \alpha_{qD}(0)S_t(2) + S_t(1)'A_{qD}(0)S_t(1) + \beta_{qD}(0)S_t(1)h_t \tag{70}$$

with $A_{qD}(0)$ and $\beta_{qD}(0)$ solved above.

8.3.3 Difference in Portfolio Euler Equation

Finally, we compute $z^D_t(1)$ from the third-order component of the difference in portfolio Euler equations (28):

$$z^D_t(1) = \frac{\left[\bar{E}^H_t er_{t+1}\right](3) - \left[E^F_t er_{t+1}\right](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)} - z^D(0)\frac{\left[E_t\left(er_{t+1}\right)^2\right](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \tag{71}$$

Here we used that $e^A_t(1) = 0$ as all agents have the same first-order expectation of the excess return, which is zero. We have already solved for $[E_t er^2_{t+1}](2)$ in (54).

Using our results, the zero-order components of the coefficients μ_{er} are:

$$\begin{aligned}
\mu_{er,1}(0) &= (1-\delta_1)\alpha_{1,qD}(0)+\delta_1 \\
\mu_{er,2}(0) &= (1-\delta_1)\alpha_{2,qD}(0)=0 \\
\mu_{er,3}(0) &= (1-\delta_1)\alpha_{5,qD}(0) \\
\mu_{er,4}(0) &= (1-\delta_1)A_{11,qD}(0)-\delta_2\alpha_{1,qA}(0)\left(1-\alpha_{1,qD}(0)\right)=0 \\
\mu_{er,5}(0) &= (1-\delta_1)A_{22,qD}(0)-\delta_2\alpha_{2,qD}(0)\left(1-\alpha_{2,qA}(0)\right)=0 \\
\mu_{er,6}(0) &= 2(1-\delta_1)A_{12,qD}+\delta_2\left((1-\alpha_{1,qD}(0))(1-\alpha_{2,qA}(0))+\alpha_{2,qD}(0)\,\alpha_{1,qA}(0)\right) \\
&= 2(1-\delta_1)A_{12,qD}+\delta_2(1-\alpha_{1,qD}(0))(1-\alpha_{2,qA}(0)) \\
\mu_{er,7}(0) &= (1-\delta_1)\beta_{1,qD}(0)+\delta_2\alpha_{1,qA}(0)\,\alpha_{5,qD}(0)=0 \\
\mu_{er,8}(0) &= (1-\delta_1)\beta_{2,qD}(0)-\delta_2(1-\alpha_{2,qA}(0))\alpha_{5,qD}(0)
\end{aligned}$$

and the first-order components of $\mu_{er,1}$ and $\mu_{er,3}$ are:

$$\begin{aligned}
\mu_{er,1}(1) &= \left[2(1-\delta_1)A_{qD,\text{row 1}}(0)+\delta_2(1-\alpha_{1,qD}(0))\frac{1}{1+\xi}\left(I_2-\omega I_4-\alpha_{qA}(0)\right)\right]\bar{S}_{t+1}(1) \\
\mu_{er,3}(1) &= \left[(1-\delta_1)\beta_{qD}(0)-\delta_2\alpha_{5,qD}(0)\,\frac{1}{1+\xi}\left(I_2-\omega I_4\right)\right]\bar{S}_{t+1}(1)
\end{aligned}$$

From (44) we then write:

$$\left[\bar{E}_t^H er_{t+1}\right](3)-\left[\bar{E}_t^F er_{t+1}\right](3)=-\frac{\sigma_{H,H}^2-\sigma_{H,F}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}\mu_{er,1}(0)\,\frac{4\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\varepsilon_{t+1}^A \tag{72}$$

The expression for $\left[E_t\left(er_{t+1}\right)^2\right](3)$ is more complex. It is useful to define:

$$\begin{aligned}
\widetilde{lin}_{t+1} &= lin_t+\mu_{er,1}\varepsilon_{t+1}^D+\mu_{er,2}\varepsilon_{t+1}^A+\mu_{er,3}h_{t+1} \\
\widetilde{quad}_{t+1} &= quad_t+\mu_{er,4}(\varepsilon_{t+1}^D)^2+\mu_{er,5}(\varepsilon_{t+1}^A)^2+\mu_{er,6}\varepsilon_{t+1}^D\varepsilon_{t+1}^A \\
&\quad+\mu_{er,7}\varepsilon_{t+1}^D h_{t+1}+\mu_{er,8}\varepsilon_{t+1}^A h_{t+1}+(1-\delta_1)\mu_{qD}h_{t+1}^2
\end{aligned}$$

so that:

$$er_{t+1}=\widetilde{lin}_{t+1}+\widetilde{quad}_{t+1}+cubic_{er}(S_t,k_{t+1}^D,k_{t+1}^A,h_t,h_{t+1},\varepsilon_{t+1}^D,\varepsilon_{t+1}^A)$$

The square of the excess return is then (dropping terms that go beyond cubic products):

$$\begin{aligned}
er_{t+1}^2 &= \left(\widetilde{lin}_{t+1}\right)^2 + 2\widetilde{lin}_{t+1}\widetilde{quad}_{t+1} \\
&= (lin_t)^2 + 2lin_t\left[\mu_{er,1}\varepsilon_{t+1}^D + \mu_{er,2}\varepsilon_{t+1}^A + \mu_{er,3}h_{t+1}\right] \\
&\quad + \left(\mu_{er,1}\varepsilon_{t+1}^D + \mu_{er,2}\varepsilon_{t+1}^A + \mu_{er,3}h_{t+1}\right)^2 \\
&\quad +2\begin{pmatrix} lin_t + \mu_{er,1}\varepsilon_{t+1}^D \\ +\mu_{er,2}\varepsilon_{t+1}^A + \mu_{er,3}h_{t+1}\end{pmatrix}\begin{pmatrix} quad_t + \mu_{er,4}(\varepsilon_{t+1}^D)^2 + \mu_{er,5}(\varepsilon_{t+1}^A)^2 + \mu_{er,6}\varepsilon_{t+1}^D\varepsilon_{t+1}^A \\ +\mu_{er,7}\varepsilon_{t+1}^D h_{t+1} + \mu_{er,8}\varepsilon_{t+1}^A h_{t+1} + (1-\delta_1)\mu_{qD}h_{t+1}^2\end{pmatrix}
\end{aligned}$$

The last row corresponds to $cubic_{er2}(S_t, k_{t+1}^D, k_{t+1}^A, h_t, h_{t+1}, \varepsilon_{t+1}^D, \varepsilon_{t+1}^A)$.

The third-order component of $E_t\left(er_{t+1}\right)^2$ is then:

$$\left[E_t er_{t+1}^2\right](3) = \left[E_t\left(\widetilde{lin}_{t+1}\right)^2\right](3) + 2\left[E_t\widetilde{lin}_{t+1}\widetilde{quad}_{t+1}\right](3) \tag{73}$$

We start with the term $\left[E_t\left(\widetilde{lin}_{t+1}\right)^2\right](3)$ in (73). Using the definition of the variance:

$$\left[E_t\left(\widetilde{lin}_{t+1}\right)^2\right](3) = \left[Var_t\left(\widetilde{lin}_{t+1}\right)\right](3) + 2\left[E_t\left(\widetilde{lin}_{t+1}\right)\right](2)\left[E_t\left(\widetilde{lin}_{t+1}\right)\right](1)$$

Note that $\left[E_t\left(\widetilde{lin}_{t+1}\right)\right](1)$ is equal to the first-order expected excess returns which are zero. We now turn to the variance term, recalling that h_{t+1} is independent from the innovations at time $t+1$:

$$\begin{aligned}
&\left[Var_t\left(\widetilde{lin}_{t+1}\right)\right](3) \\
= &\left[\left(\mu_{er,1}\right)^2 Var_t\left(\varepsilon_{t+1}^D\right)\right](3) + \left[\mu_{er,1}\mu_{er,2}Covar_t\left(\varepsilon_{t+1}^D, \varepsilon_{t+1}^A\right)\right](3) \\
&+\left[\left(\mu_{er,2}\right)^2 Var_t\left(\varepsilon_{t+1}^A\right)\right](3) + \left[\left(\mu_{er,3}\right)^2 Var_t\left(h_{t+1}\right)\right](3)
\end{aligned}$$

From the signal extraction we know that the variances only entail second-order elements. Specifically:

$$\begin{aligned}
\left[Var_t\left(\varepsilon_{t+1}^D\right)\right](2) &= \frac{4\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2 \\
\left[Var_t\left(\varepsilon_{t+1}^A\right)\right](2) &= \frac{1}{2}\sigma_a^2 \\
\left[Covar_t\left(\varepsilon_{t+1}^D, \varepsilon_{t+1}^A\right)\right](2) &= 0
\end{aligned}$$

We thus write:

$$
\begin{aligned}
\left[\left(\mu_{er,1}\right)^2 Var_t\left(\varepsilon^D_{t+1}\right)\right](3) &= \mu_{er,1}(0)\,\mu_{er,1}(1)\,\frac{8\lambda^2\theta}{1+2\lambda^2\theta}\sigma^2_a \\
\left[\left(\mu_{er,2}\right)^2 Var_t\left(\varepsilon^A_{t+1}\right)\right](3) &= \left[\mu_{er,1}\mu_{er,2}Covar_t\left(\varepsilon^D_{t+1},\varepsilon^A_{t+1}\right)\right](3)=0 \\
\left[\left(\mu_{er,3}\right)^2 Var_t\left(h_{t+1}\right)\right](3) &= 2\mu_{er,3}(0)\,\mu_{er,3}(1)\left[1+2\lambda^2\theta\right]2\sigma^2_a
\end{aligned}
$$

which gives the solution for $\left[E_t\left(\widetilde{lin}_{t+1}\right)^2\right](3)$:

$$
\begin{aligned}
\left[E_t\left(\widetilde{lin}_{t+1}\right)^2\right](3) &= \mu_{er,1}(0)\,\mu_{er,1}(1)\,\frac{8\lambda^2\theta}{1+2\lambda^2\theta}\sigma^2_a \\
&\quad +4\mu_{er,3}(0)\,\mu_{er,3}(1)\left[1+2\lambda^2\theta\right]\sigma^2_a
\end{aligned}
\tag{74}
$$

We now turn to the term $\left[E_t\widetilde{lin}_{t+1}\widetilde{quad}_{t+1}\right](3)$ in (73). It only includes cubic products, so we can focus on the zero-order components of the coefficients μ's. The relevant terms in $\widetilde{lin}_{t+1}$ and $\widetilde{quad}_{t+1}$ are then:

$$
\begin{aligned}
\widetilde{lin}_{t+1} &= lin_t+\mu_{er,1}(0)\,\varepsilon^D_{t+1}+\mu_{er,3}(0)\,h_{t+1} \\
\widetilde{quad}_{t+1} &= quad_t+\mu_{er,6}(0)\,\varepsilon^D_{t+1}\varepsilon^A_{t+1}+\mu_{er,8}(0)\,\varepsilon^A_{t+1}h_{t+1}
\end{aligned}
$$

We can then write:

$$
\begin{aligned}
&\left[E_t\widetilde{lin}_{t+1}\widetilde{quad}_{t+1}\right](3) \\
=\; & lin_t(1)\left[\begin{array}{c} quad_t(2)+\mu_{er,6}(0)\left[E_t\varepsilon^D_{t+1}\varepsilon^A_{t+1}\right](2) \\ +\mu_{er,8}(0)\left[E_t\varepsilon^A_{t+1}\right](1)\left[E_th_{t+1}\right](1)\end{array}\right] \\
& +\mu_{er,1}(0)\left[\begin{array}{c} quad_t(2)\left[E_t\varepsilon^D_{t+1}\right](1)+\mu_{er,6}(0)\,E_t\left[\left(\varepsilon^D_{t+1}\right)^2\varepsilon^A_{t+1}\right](3) \\ +\mu_{er,8}(0)\left[E_t\varepsilon^D_{t+1}\varepsilon^A_{t+1}\right](2)\left[E_th_{t+1}\right](1)\end{array}\right] \\
& +\mu_{er,3}(0)\left[\begin{array}{c} quad_t(2)\left[E_th_{t+1}\right](1)+\mu_{er,6}(0)\left[E_t\varepsilon^D_{t+1}\varepsilon^A_{t+1}\right](2)\left[E_th_{t+1}\right](1) \\ +\mu_{er,8}(0)\left[E_t\varepsilon^A_{t+1}\right](1)\left[E_t\left(h_{t+1}\right)^2\right](2)\end{array}\right]
\end{aligned}
$$

Recall that for all investors $\left[E_th_{t+1}\right](1)=0$, $\left[E_t\varepsilon^A_{t+1}\right](1)=0$, $\left[E_t\varepsilon^D_{t+1}\varepsilon^A_{t+1}\right](2)=0$. We thus get, using the fact that $lin_t(1)=-\mu_{er,1}(0)\left(1+2\lambda^2\theta\right)^{-1}h_t$:

$$
\left[E_t\widetilde{lin}_{t+1}\widetilde{quad}_{t+1}\right](3)=\mu_{er,1}(0)\,\mu_{er,6}(0)\,E_t\left[\left(\varepsilon^D_{t+1}\right)^2\varepsilon^A_{t+1}\right](3)
$$

Using the results from signal extraction in section 6, we write:

$$
\begin{aligned}
& E_t\left[\left(\varepsilon_{t+1}^D\right)^2 \varepsilon_{t+1}^A\right](3) \\
= & \frac{1}{2}\left[\begin{array}{c}
E_t\left[\left(\varepsilon_{H,t+1}\right)^3\right](3)+E_t\left[\left(\varepsilon_{F,t+1}\right)^3\right](3) \\
-E_t\left[\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2\right](3)-E_t\left[\left(\varepsilon_{H,t+1}\right)^2 \varepsilon_{F,t+1}\right](3)
\end{array}\right] \\
= & 0
\end{aligned}
$$

(73) then consists solely of (74):

$$
\left[E_t e r_{t+1}^2\right](3)=4\left[\mu_{er,1}(0)\, \mu_{er,1}(1) \frac{2\lambda^2\theta}{1+2\lambda^2\theta}+\mu_{er,3}(0)\, \mu_{er,3}(1)\left[1+2\lambda^2\theta\right]\right] \sigma_a^2 \tag{75}
$$

Using the fact that $\bar{S}_{t+1}(1)=N_1(0)\, S_t(1)+N_3(0)\, h_t$ (75) is re-written as:

$$
\left[E_t e r_{t+1}^2\right](3)=4\Omega\sigma_a^2\left[N_1(0)\, S_t(1)+N_3(0)\, h_t\right]
$$

where:

$$
\begin{aligned}
\Omega = & \mu_{er,1}(0) \frac{2\lambda^2\theta}{1+2\lambda^2\theta}\left[2(1-\delta_1) A_{qD,\text{row } 1}(0)+\delta_2(1-\alpha_{1,qD}(0)) \frac{1}{1+\xi}\left(I_2-\omega I_4\right)\right] \\
& +(1-\delta_1)\alpha_{5,qD}(0)\left[1+2\lambda^2\theta\right]\left[(1-\delta_1)\beta_{qD}(0)-\delta_2\alpha_{5,qD}(0) \frac{1}{1+\xi}\left(I_2-\omega I_4\right)\right] \\
= & \left|\begin{array}{cccc}
\Omega_1 & \Omega_2 & \Omega_3 & \Omega_4
\end{array}\right|
\end{aligned}
$$

As $A_{qD,\text{row } 1}(0)$ and $\beta_{qD}(0)$ have non-zero terms only in the second and fourth elements, we get $\Omega N_3(0)=\Omega_3 \frac{1}{\xi} \alpha_{5,qD}(0)=0$, and thus:

$$
\left[E_t e r_{t+1}^2\right](3)=4\sigma_a^2\left|\begin{array}{cccc}
0 & \rho\Omega_2+\frac{\Omega_4}{1+\xi} & 0 & \frac{1+\xi-\omega}{1+\xi}\Omega_4
\end{array}\right| S_t(1)
$$

From (63), the solution for $z_t^D(1)$ (71) gives the complete solution for $z_t^A(2)$ as well, which depends on $z_t^D(1)$. To summarize, so far we have solved for the zero, first and second-order components of q_t^D, q_t^A, k_{t+1}^D, k_{t+1}^A and z_t^A and for the zero and first-order components of z_t^D. These solutions are conditional on the noise to signal ratio λ, which we turn to solving now.

8.4 Third-order

The last element remaining to be solved, the noise to signal ratio λ, follows from equating asset supply to asset demand to the first-order. In others words, we need to equate $z_t^A(1)$ from the asset supply perspective to $z_t^A(1)$ from the portfolio demand perspective.

8.4.1 Matching asset demand and supply

$z_t^A(1)$ from the supply perspective follows from the first-order component of the difference in asset market clearing equations. This expression is in (49), which we repeat here for convenience:

$$\begin{aligned} z_t^A(1) &= \frac{1}{4}\left[\frac{1+\xi}{\xi}\alpha_{qD}(0) + I_3 - z^D(0)\left(I_1 + (1-\omega)I_3\right)\right] S_t(1) \\ &\quad + \frac{1+\xi}{4\xi}\alpha_{5,qD}(0)\, h_t \end{aligned} \tag{76}$$

An expression for $z_t^A(1)$ from the portfolio demand perspective follows from the third-order component of the average portfolio Euler equation (27), using $e_t^D(1) = 0$ and $\left[E_t(er_{t+1})^3\right](3) = 0$:

$$z_t^A(1) = \frac{\left[\bar{E}_{t+1}^A er_{t+1}\right](3) + 0.5\tau_t^D}{\left[E_t(er_{t+1})^2\right](2)} \tag{77}$$

Note that $\left[E_t(er_{t+1})^3\right](3) = 0$ as it is written as:

$$\begin{aligned} & \left[E_t(er_{t+1})^3\right](3) \\ = & \left[(E_t er_{t+1})^3 + 3(E_t er_{t+1})(var(er_{t+1}))\right](3) \\ = & \left([E_t er_{t+1}](1)\right)^3 + 3[E_t er_{t+1}](1)[var(er_{t+1})](2) \end{aligned}$$

and $[E_t er_{t+1}](1) = 0$.

Equating (76) to (77) gives:

$$\begin{aligned} & \left[E_t(er_{t+1})^2\right](2)\frac{1}{4}\left[\frac{1+\xi}{\xi}\alpha_{qD}(0) + I_3 - z^D(0)\left(I_1 + (1-\omega)I_3\right)\right] S_t(1) \\ & + \left[E_t(er_{t+1})^2\right](2)\frac{1+\xi}{4\xi}\alpha_{5,qD}(0)\, h_t \\ = & \left[\bar{E}_{t+1}^A er_{t+1}\right](3) + 0.5\tau_t^D \end{aligned} \tag{78}$$

ε_{t+1}^D and τ_t^D do not enter the first line. They enter the second line only through h_t. The only terms where ε_{t+1}^D and τ_t^D enter separately are the first two terms on the right-hand side. Recall that (using $\mu_{er,2}(0) = 0$):

$$
\begin{aligned}
& \left[\bar{E}_t^A er_{t+1}\right](3) \\
= \; & lin_t(3) + quad_t(3) \\
& + \mu_{er,1}(0) \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \left[\varepsilon_{t+1}^D - \frac{1}{1+2\lambda^2\theta} h_t\right] \frac{\lambda^2\theta}{1+2\lambda^2\theta} \sigma_a^2 \\
& + \mu_{er,1}(2) \frac{1}{1+2\lambda^2\theta} h_t + (1-\delta_1)\mu_{qD}(1) 2(1+2\lambda^2\theta)\sigma_a^2 \\
& + \mu_{er,4}(1) \frac{1}{\left(1+2\lambda^2\theta\right)^2} (h_t)^2 + \left[\mu_{er,4}(1) \frac{4\lambda^2\theta}{1+2\lambda^2\theta} + \mu_{er,5}(1) \frac{1}{2}\right] \sigma_a^2 \\
& + E_t^A cubic_{er}(.)
\end{aligned}
$$

The terms $lin_t(3)$ and $quad_t(3)$, and the coefficient $\mu_{er,1}(2)$ involve $\bar{S}_{t+1}$. As $\bar{S}_{t+1}$ only reflects terms known at time t, ε_{t+1}^D and τ_t^D do not enter there separately. Similarly, they do not enter separately in $\mu_{er,4}(1) = (1-\delta_1)A_{11,qD}(1)$ and $\mu_{er,5}(1) = (1-\delta_1)A_{22,qD}(1)$. They enter $E_t^A cubic_{er}(.)$ only through h_t. τ_t^D does not enter separately in $\left[\bar{E}_t^A er_{t+1}\right](3)$ at all. ε_{t+1}^D enters separately through the term:

$$
\mu_{er,1}(0) \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \varepsilon_{t+1}^D = \left[(1-\delta_1)\alpha_{1,qD}(0) + \delta_1\right] \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \varepsilon_{t+1}^D
$$

It follows that ε_{t+1}^D and τ_t^D enter separately in (78) through the following term on the right-hand side:

$$
\left[(1-\delta_1)\alpha_{1,qD}(0) + \delta_1\right] \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \frac{\lambda^2\theta}{1+2\lambda^2\theta} \sigma_a^2 \varepsilon_{t+1}^D + \frac{\tau_t^D}{2}
$$

ε_{t+1}^D and τ_t^D can only enter in the same combination as in $h_t = \varepsilon_{t+1}^D + \lambda\tau_t^D/\tau$ as they only enter elsewhere in (78) through h_t. It follows that

$$
\left[(1-\delta_1)\alpha_{1,qD}(0) + \delta_1\right] \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \frac{\lambda^2\theta}{1+2\lambda^2\theta} \sigma_a^2 = \frac{1}{2} \frac{\tau}{\lambda}
$$

so that

$$
\frac{1+2\lambda^2\theta}{2\lambda^2\theta} \frac{\tau}{\sigma_a^2} = \left[(1-\delta_1)\alpha_{1,qD}(0) + \delta_1\right] \frac{\sigma_{H,H}^2 + \sigma_{H,F}^2}{\sigma_{H,H}^2 \sigma_{H,F}^2} \lambda
$$

which is the implicit solution for λ.

While zero, first and second-order components are sufficient for most purposes, after having solved for λ we can also compute $q_t^D(3)$ from (78). From our conjecture for the relative asset price we have

$$\begin{aligned} q_t^D(3) &= \alpha_{qD}(0)S_t(3) + \alpha_{qD}(1)S_t(2) + \alpha_{qD}(2)S_t(1) + \alpha_{5,qD}(2)h_t + \\ &\quad 2S_t(1)'A_{qD}(0)S_t(2) + S_t(1)'A_{qD}(1)S_t(1) + \beta_{qD}(1)S_t(1)h_t + \\ &\quad \beta_{qD}(0)S_t(2)h_t + \mu_{qD}(1)h_t^2 + cubic(S_t, h_t) \end{aligned} \tag{79}$$

The parameters that remain to be solved are $\alpha_{qD}(2)$, $\alpha_{5,qD}(2)$, $A_{qD}(1)$, $\beta_{qD}(1)$, $\mu_{qD}(1)$ and zero-order components of the 35 parameters associated with the cubic terms. These coefficients enter the expression for $[\bar{E}_t^A er_{t+1}](3)$, which enters (78). Setting the coefficients on the cubic terms in h_t and S_t in (78) equal to zero allows us to compute these coefficients.

9 Individual portfolio shares

The individual zero-order portfolio shares are computed by taking the second-order component of (25)-(26) for individual investors:

$$\begin{aligned} z_{Hj}(0) &= \frac{1}{2} + \frac{\left[E_t^{H,j} er_{t+1}\right](2) + \tau}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \\ z_{Fj}(0) &= \frac{1}{2} + \frac{\left[E_t^{F,j} er_{t+1}\right](2) - \tau}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \end{aligned}$$

where $\left[E_t\left(er_{t+1}\right)^2\right](2)$ is the same for all investors and is given by (54).

The second-order components of expected excess returns are given by taking the individual investor's expectation of (35). From the signal extraction, we know that all agents worldwide have the same expectations for the second-order component of the expectation of cross-product of innovation, as well as the first-order component of expected innovation. Individual investors disagree on the second-order component of expected innovations. We therefore write:

$$\begin{aligned} \left[E_t^{H,j} er_{t+1}\right](2) &= \left[\bar{E}_t^A er_{t+1}\right](2) + \mu_{er,1}(0)\left[E_t^{H,j}\varepsilon_{t+1}^D\right](2) + \mu_{er,2}(0)\left[E_t^{H,j}\varepsilon_{t+1}^A\right](2) \\ \left[E_t^{F,j} er_{t+1}\right](2) &= \left[\bar{E}_t^A er_{t+1}\right](2) + \mu_{er,1}(0)\left[E_t^{F,j}\varepsilon_{t+1}^D\right](2) + \mu_{er,2}(0)\left[E_t^{F,j}\varepsilon_{t+1}^A\right](2) \end{aligned}$$

Using $\mu_{er,2}(0) = 0$, $\left[\bar{E}_t^A er_{t+1}\right](2) = 0$ and the results from the signal extraction, we write:

$$
\begin{aligned}
\left[E_t^{H,j} er_{t+1}\right](2) &= \mu_{er,1}(0)\left[\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2} - \frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right]\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2 \\
\left[E_t^{F,j} er_{t+1}\right](2) &= \mu_{er,1}(0)\left[\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2} - \frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right]\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2
\end{aligned}
$$

The individual portfolio shares are then:

$$
z_{Hj}(0) = \frac{1+z^D(0)}{2} + \frac{\mu_{er,1}(0)}{\left[E_t\left(er_{t+1}\right)^2\right](2)}\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\left[\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2} - \frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right] \tag{80}
$$

$$
z_{Fj}(0) = \frac{1-z^D(0)}{2} + \frac{\mu_{er,1}(0)}{\left[E_t\left(er_{t+1}\right)^2\right](2)}\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\left[\frac{\epsilon_{j,t}^{F,H}}{\sigma_{H,F}^2} - \frac{\epsilon_{j,t}^{F,F}}{\sigma_{H,H}^2}\right] \tag{81}
$$

The cross-sectional variance of portfolio shares within a country is then:

$$
\begin{aligned}
&\int \left(z_{Hj}(0) - z_H(0)\right)^2 dj \\
&= \left[\frac{\mu_{er,1}(0)}{\left[E_t\left(er_{t+1}\right)^2\right](2)}\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\right]^2 \int \left(\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2} - \frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right)^2 dj \\
&= \left[\frac{\mu_{er,1}(0)}{\left[E_t\left(er_{t+1}\right)^2\right](2)}\frac{2\lambda^2\theta}{1+2\lambda^2\theta}\sigma_a^2\right]^2 \frac{\sigma_{H,F}^2+\sigma_{H,H}^2}{\sigma_{H,H}^2\sigma_{H,F}^2}
\end{aligned} \tag{82}
$$

10 Euler equation errors

This section describes the procedure we follow to compute numerically the portfolio Euler equation errors for agent j in country H. We use the first-order approximations to compute the Euler errors. We compute the errors over the entire ergodic state by simulating the model over 100,000 periods. We then compute the largest absolute error and the root mean squared error. We assume that for agent j the cost $\tau_{Hj,t}$ is always equal to the average $\tau_{H,t}$ for country H. Recall that we assume that the private component of the transaction cost is so noisy that investors do not make inferences based on it. We thus set $\tau_{Hj,t} = \tau_{H,t}$ in our computations for simplicity. We consider both the case where the agent receives signals about next period's productivity in both countries with errors drawn from the distribution of the errors and a representative agent case who always receives the average private signal.

The expectation by agent j at time t is conditional on S_t, h_t and signals $v_{j,t}^{H,H}$, $v_{j,t}^{H,F}$. The unknowns at $t+1$ are $\varepsilon_{H,t+1}$, $\varepsilon_{F,t+1}$, ε_{t+2}^D, and τ_{t+1}^D. The distribution of $\varepsilon_{H,t+1}$ and $\varepsilon_{F,t+1}$ follows from signal extraction and is specific to agents j; it depends on the private signals of the agent. The distributions of ε_{t+2}^D and τ_{t+1}^D are the unconditional distributions of these variables as no additional information about them is available at time t.

In computing expectations we discretize the normal distribution of the 5 stochastic variables. To do so for a $N(0,1)$ distribution, we first break the values from $-\infty$ to $+\infty$ into N intervals that have an equal density of $1/N$ based on the standard continuous $N(0,1)$ distribution. The discretized variable can then take on N values that each have probability $1/N$. The value in the middle is 0. The negative values are symmetric to the positive values. The positive values, with the exception of the largest, are set equal to the expectation of the continuous normal distribition conditional on being in the corresponding interval. The largest positive and negative values are set such that the standard deviation of the discretized variable is 1.

For ε_{t+2}^D and τ_{t+1}^D we multiply the values of the discretized $N(0,1)$ distribution by their unconditional standard deviations. For the vector $(\varepsilon_{H,t+1}, \varepsilon_{F,t+1})'$ we start from the conditional continuous distribution that is normal with a mean and variance that we write compactly as μ^j and Ω. Since Ω is symmetric we can diagonolize it as $\Omega = P\Delta P'$ where Δ is a diagonal matrix containing the eigenvalues of Ω. If x is a vector of two independent $N(0,1)$ distributions, the conditional distribution of $P\Delta^{1/2}x + \mu^j$ corresponds to the conditional distribution of $(\varepsilon_{H,t+1}, \varepsilon_{F,t+1})'$. We therefore apply this transformation to a vector of two independent discretized $N(0,1)$ distributions. Using these discretized distributions we can then compute expectations based on a finite number of possible values of the unknowns.

The Home return in the portfolio expression depends on $a_{H,t+1}$, $k_{H,t+1}$, $q_{H,t}$ and $q_{H,t+1}$. We use the sum of their zero and first-order components. The same is done for the Foreign return. The overall portfolio return in addition depends on $z_{Hj,t}$. Its zero order component is given in (80). We also need its first-order component, which we have not yet computed above.

In order to compute the first-order component of $z_{Hj,t}$ we start with the portfolio Euler equation for agent j in (25), which we repeat here for convenience:

$$
\begin{aligned}
0 \quad = \quad & E_t^{H,j}\left(er_{t+1}\right) + \tau_{Hj,t} - \frac{2z_{Hj}(0)-1}{2} E_t^{H,j}\left(er_{t+1}\right)^2 \\
& -(2z_{Hj}(0)-1)\tau_{Hj,t} E_t^{H,j} er_{t+1} - z_{Hj,t} E_t^{H,j}\left(er_{t+1}\right)^2 \\
& + \left[\frac{1}{6} - z_{Hj}(0)\left(1 - z_{Hj}(0)\right)\right] E_t^{H,j}\left(er_{t+1}\right)^3
\end{aligned}
$$

We know from section 7.2 that the expectation of er_{t+1}^2 and er_{t+1}^3 is the same across the agents to all relevant orders, so that we can remove the j superscript from these expectations.

We take the third-order component of (25). We know from section 8.2.1 that expected excess returns are zero to a first order for all agents: $\left[E_t^{H,j} er_{t+1}\right](1) = 0$, and from section 8.4.1 that $\left[E_t^{H,j}\left(er_{t+1}\right)^3\right](3) = 0$. The third-order component of (25) is then:

$$
z_{Hj,t}(1) = \frac{[E_t^{H,j} er_{t+1}](3) + \tau_{H,t}(3) + (0.5 - z_{H,j}(0))[E_t^{H,j} er_{t+1}^2](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \tag{83}
$$

where we used our assumption that $\tau_{Hj,t} = \tau_{H,t}$. The only expression here that we have not yet computed is $[E_t^{H,j} er_{t+1}](3)$.

We can write the expected excess return as:

$$
\begin{aligned}
& [E_t^{H,j} er_{t+1}](3) = \Big([E_t^{H,j} er_{t+1}](3) - [\bar{E}_t^{H} er_{t+1}](3)\Big) + \\
& 0.5\left([\bar{E}_t^{H} er_{t+1}](3) - [\bar{E}_t^{F} er_{t+1}](3)\right) + [\bar{E}_t^{A}(er_{t+1})](3)
\end{aligned} \tag{84}
$$

$[\bar{E}_t^{H} er_{t+1}](3) - [\bar{E}_t^{F} er_{t+1}](3)$ is given by (72). Using (77) we have

$$
[\bar{E}_t^{A}(er_{t+1})](3) = [E_t er_{t+1}^2](2) z_t^A(1) - 0.5\tau_t^D \tag{85}
$$

with $z_t^A(1)$ given by (49).

All that is left is to compute the difference between $[E_t^{H,j} er_{t+1}](3)$ and $[\bar{E}_t^{H} er_{t+1}](3)$. To this end we start from (35). There is disagreement among investors (due to private signals) only with regards to the second-order component of the expectation of ε_{t+1}^D and ε_{t+1}^A and the third-order component of the expectation of $(\varepsilon_{t+1}^D)^2$, $(\varepsilon_{t+1}^A)^2$ and $\varepsilon_{t+1}^D\varepsilon_{t+1}^A$. There is no disagreement with regards to cubic terms in these productivity innovations (at least up to third-order). As a result there is also no disagreement with regards to the cubic terms in (35), up to third-order.

Limiting ourselves to the terms in (35) about which there is disagreement among investors, and taking the third-order component of the expectation for agent j, we have:

$$\begin{aligned}
&\mu_{er,1}(1)[E_t^{H,j}\varepsilon_{t+1}^D](2)+\mu_{er,2}(1)[E_t^{H,j}\varepsilon_{t+1}^A](2)\\
&+\mu_{er,4}(0)\,[E_t^{H,j}(\varepsilon_{t+1}^D)^2](3)+\mu_{er,5}(0)\,[E_t^{H,j}(\varepsilon_{t+1}^A)^2](3)+\mu_{er,6}(0)[E_t^{H,j}\varepsilon_{t+1}^D\varepsilon_{t+1}^A](3)\\
&+\left[\mu_{er,7}(0)\,[E_t^{H,j}\varepsilon_{t+1}^D](2)+\mu_{er,8}(0)\,[E_t^{H,j}\varepsilon_{t+1}^A](2)\right]h_{t+1}(1)
\end{aligned}$$

From section 8.3.3 we know that $\mu_{er,4}(0)=\mu_{er,5}(0)=\mu_{er,7}(0)=0$. h_{t+1} consists of the productivity innovations at time $t+2$ and portfolio costs at time $t+1$, which are all independent from ε_{t+1}^D and ε_{t+1}^A so the last term is zero. The expression thus simplifies to:

$$\mu_{er,1}(1)[E_t^{H,j}\varepsilon_{t+1}^D](2)+\mu_{er,2}(1)[E_t^{H,j}\varepsilon_{t+1}^A](2)+\mu_{er,6}(0)[E_t^{H,j}\varepsilon_{t+1}^D\varepsilon_{t+1}^A](3) \tag{86}$$

We have already shown that:

$$\begin{aligned}
\mu_{er,1}(1) &= (1-\delta_1)\,\alpha_{1,qD}(1)+2(1-\delta_1)A_{qD,\text{row }1}(0)\,\bar{S}_{t+1}(1)\\
&\quad+\delta_2(1-\alpha_{1,qD}(0))\,(I_2-\omega I_4-\alpha_{qA}(0))\,\bar{S}_{t+1}(1)\\
\mu_{er,2}(1) &= 2(1-\delta_1)A_{qD,\text{row }2}(0)\,\bar{S}_{t+1}(1)+\delta_2(1-\alpha_{2,qA}(0))\,(I_1-\omega I_2-\alpha_{qD}(0))\,\bar{S}_{t+1}(1)\\
\mu_{er,6}(0) &= 2(1-\delta_1)A_{12,qD}(0)+\delta_2(1-\alpha_{1,qD}(0))(1-\alpha_{2,qA}(0))
\end{aligned}$$

In terms of expectations of the innovations, we write:

$$[E_t^{Hj}\varepsilon_{t+1}^D](2) = \left(\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}-\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right)\frac{2\lambda^2\theta\sigma_a^2}{1+2\lambda^2\theta} \tag{87}$$

$$[E_t^{Hj}\varepsilon_{t+1}^A](2) = \left(\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}+\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right)\frac{\sigma_a^2}{2} \tag{88}$$

$$\begin{aligned}
[E_t^{Hj}\varepsilon_{t+1}^D\varepsilon_{t+1}^A](3) &= \frac{1}{2}\left(\left[E_t^{j,H}(\varepsilon_{H,t+1})^2\right](3)-\left[E_t^{j,H}(\varepsilon_{F,t+1})^2\right](3)\right)\\
&= \left(\frac{\epsilon_{j,t}^{H,H}}{\sigma_{H,H}^2}+\frac{\epsilon_{j,t}^{H,F}}{\sigma_{H,F}^2}\right)\frac{\sigma_a^2}{2\left(1+2\lambda^2\theta\right)}h_t
\end{aligned} \tag{89}$$

(86) then give us the difference between $[E_t^{H,j}er_{t+1}](3)$ and $[\bar{E}_t^{H}er_{t+1}](3)$, which is the final element needed for the first-order component of $z_t^{H,j}(1)$.

11 Balance of payments accounting

11.1 National savings and investment

In period t the old Home agents enter the period with the following quantities of equities (we abstract from j indexes as we focus on first-order aggregates):

$$G^H_{H,t-1} = \frac{z_{H,t-1}\left(W_{H,t-1} - C^H_{y,t-1}\right)}{Q_{H,t-1}} \quad ; \quad G^H_{F,t-1} = \frac{\left(1 - z_{H,t-1}\right)\left(W_{H,t-1} - C^H_{y,t-1}\right)}{Q_{F,t-1}}$$

The consumption of old agents is the total return on their portfolio. The income of old agents is the dividend stream they receive, while capital gains and losses are not counted as income streams in national accounts. We also consider that income is net of depreciation. The savings of the old Home agents are then (we ignore the iceberg cost on holdings of Foreign equity as it represents a source of income for intermediaries that is fully consumed, and thus does not affect savings):

$$\begin{aligned}
S^H_{o,t} &= \left[(1-\omega) A_{Ht} \left(K_{H,t}\right)^{-\omega} - \delta Q_{H,t}\right] G^H_{H,t-1} \\
&\quad + \left[(1-\omega) A_{Ft} \left(K_{F,t}\right)^{-\omega} - \delta Q_{F,t}\right] G^H_{F,t-1} \\
&\quad - \left(R_{H,t} Q_{H,t-1} G^H_{H,t-1} + R_{F,t} Q_{F,t-1} G^H_{F,t-1}\right) \\
&= -Q_{H,t} G^H_{H,t-1} - Q_{F,t} G^H_{F,t-1} \\
&= -\left[z_{H,t-1} \frac{Q_{H,t}}{Q_{H,t-1}} + \left(1 - z_{H,t-1}\right) \frac{Q_{F,t}}{Q_{F,t-1}}\right] \left(W_{H,t-1} - C^H_{y,t-1}\right)
\end{aligned}$$

The dissavings by old agents reflects the liquidation value of their portfolio. National savings in the Home country are:

$$\begin{aligned}
S^H_t &= S^H_{y,t} + S^H_{o,t} \\
&= W_{H,t} - C^H_{y,t} - \left[z_{H,t-1} \frac{Q_{H,t}}{Q_{H,t-1}} + \left(1 - z_{H,t-1}\right) \frac{Q_{F,t}}{Q_{F,t-1}}\right] \left(W_{H,t-1} - C^H_{y,t-1}\right) \qquad (90)
\end{aligned}$$

Investment in the national accounts is also defined as net of depreciation:

$$I^{net}_{H,t} = I_{H,t} - \delta K_{H,t} = K_{H,t+1} - K_{H,t} \qquad (91)$$

The corresponding relation for the Foreign country are:

$$\begin{aligned}
S^F_t &= W_{F,t} - C^F_{y,t} - \left[z_{F,t-1} \frac{Q_{H,t}}{Q_{H,t-1}} + \left(1 - z_{F,t-1}\right) \frac{Q_{F,t}}{Q_{F,t-1}}\right] \left(W_{F,t-1} - C^F_{y,t-1}\right) \qquad (92) \\
I^{net}_{F,t} &= I_{F,t} - \delta K_{F,t-1} \qquad (93)
\end{aligned}$$

Using (5) and (6), the values of world savings and investment are equal: $S_t^H + S_t^F = Q_{H,t} I_{H,t}^{net} + Q_{F,t} I_{F,t}^{net}$.

The first-order component of (90) is written as:

$$
\begin{aligned}
s_t^H(1) &= \Delta w_{H,t}(1) - z_H(0)\Delta q_{H,t}(1) - (1 - z_H(0))\Delta q_{F,t}(1) \\
&= \Delta a_{H,t}(1) + (1-\omega)\Delta k_{H,t}(1) - z_H(0)\Delta q_{H,t}(1) - (1 - z_H(0))\Delta q_{F,t}(1)
\end{aligned}
$$

where $s_t^H(1) = S_t^H(1)/W^F(0)$, $W^F(0) = \beta(1+\beta)^{-1} W_H(0)$ is the zero-order component of financial wealth, $w_{H,t}(1) = W_{H,t}(1)/W^F(0)$ and $\Delta g_t(1) = g_t(1) - g_{t-1}(1)$. This is re-written as:

$$
\begin{aligned}
s_t^H(1) &= \left[\Delta a_t^A(1) + (1-\omega)\Delta k_t^A(1)\right] + \frac{1}{2}\left[\Delta a_t^D(1) + (1-\omega)\Delta k_t^D(1)\right] \\
&\quad - \Delta q_t^A(1) - \frac{z^D(0)}{2}\Delta q_t^D(1) \\
&= \alpha_{sH}(0)\Delta S_t(1) - \frac{z^D(0)}{2}\Delta q_t^D(1) \qquad (94)
\end{aligned}
$$

where:

$$
\alpha_{sH}(0) = \frac{1}{1+\xi} I_2 + \frac{1+\xi-\omega}{1+\xi} I_4 + \frac{1}{2}\left[I_1 + (1-\omega) I_3\right]
$$

Similarly, the first-order component of (92) is:

$$
s_t^F(1) = \alpha_{sF}(0)\Delta S_t(1) + \frac{z^D(0)}{2}\Delta q_t^D(1) \qquad (95)
$$

where:

$$
\alpha_{sF}(0) = \frac{1}{1+\xi} I_2 + \frac{1+\xi-\omega}{1+\xi} I_4 - \frac{1}{2}\left[I_1 + (1-\omega) I_3\right]
$$

Taking the difference between (94) and (95) we get:

$$
s_t^D(1) = \left[I_1 + (1-\omega) I_3\right]\Delta S_t(1) - z^D(0)\Delta q_t^D(1) \qquad (96)
$$

Using (13), the first-order component of (91) is:

$$
i_{H,t}^{net}(1) = \Delta k_{H,t+1}(1) = \frac{1}{\xi} q_{H,t}(1) \qquad (97)
$$

where $i_{H,t}^{net}(1) = I_{H,t}^{net}(1) / \exp[k(0)]$. Similarly, the first-order component of (93) is:

$$
i_{F,t}^{net}(1) = \frac{1}{\xi} q_{F,t}(1) \qquad (98)
$$

11.2 Capital flows

A useful measure is the passive portfolio share that combines the steady-state holdings of quantities of assets with the actual asset prices. For Home investors, we write:

$$z^p_{H,t} = \frac{z_H(0)\exp[q_{H,t}]}{z_H(0)\exp[q_{H,t}] + (1 - z_H(0))\exp[q_{F,t}]}$$

The first-order passive portfolio share is the same for all investors:

$$z^p_t(1) = \frac{1 - \left(z^D(0)\right)^2}{4} q^D_t(1) \tag{99}$$

Taking the first-order component of (21) we write:

$$q^D_t(1) + k^D_{t+1}(1) = z^D(0)\left[I_1 + (1-\omega) I_3\right] S_t(1) + 4 z^A_t(1)$$

Next, we take the difference between this relation and its lagged value and use (96) and (99) to obtain ($i^{D,net}_t(1) = \Delta k^D_{t+1}(1)$):

$$i^{D,net}_t(1) - z^D(0) s^D_t(1) = 4\left[\Delta z^A_t(1) - \Delta z^p_t(1)\right]$$

Gross outflows and inflows are the change in the value of cross-border asset holdings, evaluated at current asset prices:

$$\begin{aligned}
OUTFLOWS_t &= Q_{F,t}\left(G^H_{F,t} - G^H_{F,t-1}\right) \\
&= (1 - z_{H,t})\left(W_{H,t} - C^H_{y,t}\right) - \frac{Q_{F,t}}{Q_{F,t-1}}(1 - z_{H,t-1})\left(W_{H,t-1} - C^H_{y,t-1}\right) \\
INFLOWS_t &= z_{F,t}\left(W_{F,t} - C^F_{y,t}\right) - \frac{Q_{H,t}}{Q_{H,t-1}} z_{F,t-1}\left(W_{F,t-1} - C^F_{y,t-1}\right)
\end{aligned}$$

The first-order components of gross outflows is:

$$\begin{aligned}
outflows_t(1) &= -\Delta z_{H,t}(1) + (1 - z_H(0))\left[\Delta a_{H,t}(1) + (1-\omega)\Delta k_{H,t}(1)\right] \\
&\quad - (1 - z_H(0))\Delta q_{F,t}(1)
\end{aligned}$$

where: $outflows_t(1) = OUTFLOWS_t(1)/W^F(0)$. Using (94) and (99) we write:

$$\begin{aligned}
& outflows_t(1) \\
= \; & (1 - z_H(0)) s^H_t(1) - \left[\Delta z^A_t(1) - \Delta z^p_t(1)\right] - \frac{1}{2}\Delta z^D_t(1) \\
= \; & (1 - z_H(0)) s^H_t(1) + \frac{z^D(0)}{2}\frac{\Delta\left[E_t(er_{t+1})^2\right](3)}{\left[E_t(er_{t+1})^2\right](2)} \\
& - \frac{\Delta \bar{E}_t er_{t+1}(3)^{IS}}{\left[E_t(er_{t+1})^2\right](2)} - \frac{1}{2}\frac{\Delta\left[\bar{E}^H_t er_{t+1}\right](3) - \Delta\left[E^F_t er_{t+1}\right](3)}{\left[E_t(er_{t+1})^2\right](2)}
\end{aligned} \tag{100}$$

where we used (71) and we defined:

$$\Delta \bar{E}_t er_{t+1}(3)^{IS} = \frac{\left[E_t\left(er_{t+1}\right)^2\right](2)}{4}\left[i_t^{D,net}(1) - z^D(0)\, s_t^D(1)\right]$$

Similarly, the first-order component of gross inflows is:

$$\begin{aligned}
& inflows_t(1) \\
= \;& z_F(0)\, s_t^F(1) + \left[\Delta z_t^A(1) - \Delta z_t^p(1)\right] - \frac{1}{2}\Delta z_t^D(1) \\
= \;& (1 - z_H(0))\, s_t^F(1) + \frac{z^D(0)}{2}\frac{\Delta\left[E_t\left(er_{t+1}\right)^2\right](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)} \\
& + \frac{\Delta \bar{E}_t er_{t+1}(3)^{IS}}{\left[E_t\left(er_{t+1}\right)^2\right](2)} - \frac{1}{2}\frac{\Delta\left[\bar{E}_t^H er_{t+1}\right](3) - \Delta\left[\bar{E}_t^F er_{t+1}\right](3)}{\left[E_t\left(er_{t+1}\right)^2\right](2)}
\end{aligned} \tag{101}$$

Combining (100) and (101) we write:

$$\begin{aligned}
outflows_t(1) - inflows_t(1) &= \frac{1}{2}\left[s_t^D(1) - i_t^{D,net}(1)\right] \\
outflows_t(1) + inflows_t(1) &= \left(1 - z^D(0)\right) s_t^A(1) - \Delta z_t^D(1)
\end{aligned}$$

12 Public Information

In this section we describe the solution when instead of private signals there are common public signals

$$v_t^H = \varepsilon_{H,t+1} + \epsilon_t^H \tag{102}$$

$$v_t^F = \varepsilon_{F,t+1} + \epsilon_t^F \tag{103}$$

with $var(\epsilon^{H,t}) = var(\epsilon_t^F) = \sigma_v^2$. This implies

$$E_t \varepsilon_{H,t+1} = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_v^2} v_t^H \tag{104}$$

$$E_t \varepsilon_{F,t+1} = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_v^2} v_t^F \tag{105}$$

$\varepsilon_{H,t+1}$ and $\varepsilon_{F,t+1}$ remain uncorrelated. They have a variance of

$$var(\varepsilon_{H,t+1}) = var(\varepsilon_{F,t+1}) = \frac{\sigma_a^2 \sigma_v^2}{\sigma_a^2 + \sigma_v^2} \tag{106}$$

In this case it makes sense to assume that the errors in the signals are second-order, for three reasons. First, this is consistent with the other public signal with variance σ_a^2 that is also second-order. Second, making it zero-order is pointless as it will then receive zero weight to the orders that we care about. The results will then be as if this signal did not exist. Third, it is of interest to compare to the private information case when the public signal leads to the same conditional variance (second-order component) as in the case of private information. This requires that σ_v^2 is second-order. If it were zero-order the second-order component of the conditional variance is σ_a^2, which is larger than under private information.

Assuming therefore that σ_v^2 is second-order, the variance of $\varepsilon_{H,t+1}$ and $\varepsilon_{F,t+1}$ only has a second-order component. The expectations only have a first-order component. We also have

$$E_t\varepsilon_{t+1}^D = \frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^D \tag{107}$$

$$E_t\varepsilon_{t+1}^A = \frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^A \tag{108}$$

which only have first-order components. Here $v_t^D = v_t^H - v_t^F$ and $v_t^A = 0.5(v_t^H + v_t^F)$.

We have

$$E_t\varepsilon_{H,t+1}^2 = \left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 \left(v_t^H\right)^2 + \frac{\sigma_a^2\sigma_v^2}{\sigma_a^2+\sigma_v^2} \tag{109}$$

$$E_t\varepsilon_{F,t+1}^2 = \left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 \left(v_t^F\right)^2 + \frac{\sigma_a^2\sigma_v^2}{\sigma_a^2+\sigma_v^2} \tag{110}$$

which only has a second-order component. We also have

$$E_t(\varepsilon_{t+1}^D)^2 = \left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 \left(v_t^D\right)^2 + 2\frac{\sigma_a^2\sigma_v^2}{\sigma_a^2+\sigma_v^2} \tag{111}$$

$$E_t(\varepsilon_{t+1}^A)^2 = \left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 \left(v_t^A\right)^2 + \frac{1}{2}\frac{\sigma_a^2\sigma_v^2}{\sigma_a^2+\sigma_v^2} \tag{112}$$

$$E_t\varepsilon_{t+1}^A\varepsilon_{t+1}^D = \left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 v_t^D v_t^A \tag{113}$$

which again only have second-order components.

The asset price conjectures (30) and (31) now become

$$q_t^D = \alpha_{qD}S_t + S_t'A_{qD}S_t \tag{114}$$

$$q_t^A = \alpha_{qA}S_t + S_t'A_{qA}S_t \tag{115}$$

where $S_t = (a_t^D, a_t^A, k_t^D, k_t^A, v_t^D, v_t^A)'$.

Let $\bar{S}_{t+1}$ again be the expected component of S_{t+1}, which is $(\rho a_t^D, \rho a_t^A, k_{t+1}^D, k_{t+1}^A, 0, 0)'$. Then

$$S_{t+1} = \bar{S}_{t+1} + \begin{pmatrix} \varepsilon_{t+1}^D \\ \varepsilon_{t+1}^A \\ 0 \\ 0 \\ v_{t+1}^D \\ v_{t+1}^A \end{pmatrix} \tag{116}$$

It will also be useful to compute an expression of $S'_{t+1}AS_{t+1}$ for a symmetric 6 by 6 matrix A. We take the expectation right away, which condenses notation as a lot of terms have expectation 0: terms proportional in v_{t+1}^D, v_{t+1}^A and the product $v_{t+1}^D v_{t+1}^A$. We have

$$\begin{aligned} E_t S'_{t+1}AS_{t+1} &= \bar{S}'_{t+1}A\bar{S}_{t+1} + 2A_{row\ 1}\bar{S}_{t+1}E_t\varepsilon_{t+1}^D + 2A_{row\ 2}\bar{S}_{t+1}E_t\varepsilon_{t+1}^A + \\ & A_{11}E_t(\varepsilon_{t+1}^D)^2 + A_{22}E_t(\varepsilon_{t+1}^A)^2 + A_{55}E_t(v_{t+1}^D)^2 + A_{66}E_t(v_{t+1}^A)^2 + 2A_{12}E_t\varepsilon_{t+1}^A\varepsilon_{t+1}^D \end{aligned} \tag{117}$$

We will need an expression for the first and second-order components of the expected excess return. Limiting ourselves to the linear and quadratic terms, which is sufficient, the excess return is

$$er_{t+1} = (1-\delta_1)q_{t+1}^D - q_t^D + \delta_1(a_{t+1}^D - \omega k_{t+1}^D) + \delta_2(a_{t+1}^A - \omega k_{t+1}^A - q_{t+1}^A)(a_{t+1}^D - \omega k_{t+1}^D - q_{t+1}^D) \tag{118}$$

Using the results above, the expected excess return can be written as

$$\begin{aligned} E_t er_{t+1} &= lin_t + quad_t + \mu_{er,1}E_t\varepsilon_{t+1}^D + \mu_{er,2}E_t\varepsilon_{t+1}^A + \mu_{er,4}E_t(\varepsilon_{t+1}^D)^2 \\ &+ \mu_{er,5}E_t(\varepsilon_{t+1}^A)^2 + \mu_{er,6}E_t\varepsilon_{t+1}^D\varepsilon_{t+1}^A + \mu_{er,9}E_t(v_{t+1}^D)^2 + \mu_{er,10}E_t(v_{t+1}^A)^2 \end{aligned} \tag{119}$$

where

$$\begin{aligned}
\mu_{er,1} &= (1-\delta_1)\alpha_{1,qD}+\delta_1+2(1-\delta_1)A_{qD,\text{row }1}\bar{S}_{t+1} \\
&+\delta_2\left[\left(1-\alpha_{1,qD}\right)\left(I_2-\omega I_4-\alpha_{qA}\right)-\alpha_{1,qA}\left(I_1-\omega I_3-\alpha_{qD}\right)\right]\bar{S}_{t+1} \\
\mu_{er,2} &= (1-\delta_1)\alpha_{2,qD}+2(1-\delta_1)A_{qD,\text{row }2}\bar{S}_{t+1} \\
&-\delta_2\left[\alpha_{2,qD}\left(I_2-\omega I_4-\alpha_{qA}\right)-\left(1-\alpha_{2,qA}\right)\left(I_1-\omega I_3-\alpha_{qD}\right)\right]\bar{S}_{t+1} \\
\mu_{er,4} &= (1-\delta_1)A_{11,qD}-\delta_2\alpha_{1,qA}\left(1-\alpha_{1,qD}\right) \\
\mu_{er,5} &= (1-\delta_1)A_{22,qD}-\delta_2\alpha_{2,qD}\left(1-\alpha_{2,qA}\right) \\
\mu_{er,6} &= 2(1-\delta_1)A_{12,qD}+\delta_2\left((1-\alpha_{1,qD})(1-\alpha_{2,qA})+\alpha_{2,qD}\alpha_{1,qA}\right) \\
\mu_{er,9} &= \delta_2\alpha_{5,qD}\alpha_{5,qA}+(1-\delta_1)A_{55,qD} \\
\mu_{er,10} &= \delta_2\alpha_{6,qD}\alpha_{6,qA}+(1-\delta_1)A_{66,qD}
\end{aligned}$$

and

$$\begin{aligned}
lin_t &= \left[(1-\delta_1)\alpha_{qD}+\delta_1\left(I_1-\omega I_3\right)\right]\bar{S}_{t+1}-\alpha_{qD}S_t \\
quad_t &= (1-\delta_1)\bar{S}'_{t+1}A_{qD}\bar{S}_{t+1}-S'_tA_{qD}S_t \\
&+\delta_2\bar{S}'_{t+1}\left(I_1-\omega I_3-\alpha_{qD}\right)'\left(I_2-\omega I_4-\alpha_{qA}\right)\bar{S}_{t+1}
\end{aligned}$$

The coefficients $\mu_{er,1}$,$\mu_{er,2}$,$\mu_{er,4}$,$\mu_{er,5}$ and $\mu_{er,6}$ are the same as before.

We now impose the order components of the equations, relying to a large extent on the results that we already have. Imposing first-order components of the average capital accumulation equation and the average market clearing condition gives the same expressions for $\alpha_{qA}(0)$ and $\alpha_{kA}(0)$ as before. Note that the last two elements of these vectors, which did not exist before, are zero. The expressions for $k^D_{t+1}(1)$ and $z^A_t(1)$, based on the difference in capital accumulation equations and the difference in asset market clearing equations, also remain the same, with the terms in h_t dropped. These depend on $\alpha_{qD}(0)$, which now changes a bit. Imposing $E_t er_{t+1}(1)=0$ we have

$$lin_t(1)+\mu_{er,1}(0)\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^D+\mu_{er,2}(0)\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^A=0 \tag{120}$$

It is easily seen that this implies the same first 4 elements of $\alpha_{qD}(0)$ as before. Since $\alpha_{2,qD}(0)=0$ this also implies $\mu_{er,2}(0)=0$. This implies that $\alpha_{6,qD}(0)=0$. Finally, we have

$$\alpha_{5,qD}(0)=\frac{\left[(1-\delta_1)\alpha_{1,qD}(0)+\delta_1\right]}{1+\left[\delta_1\omega-(1-\delta_1)\alpha_{3,qD}(0)\right]\frac{1}{\xi}}\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2} \tag{121}$$

Taking the second-order component of the difference in portfolio Eulers we get the same expression for $z^D(0)$ as before, but the expression of the second-order component of the variance of the excess return becomes

$$[var(er_{t+1})](2) = 2\left[(1-\delta_1)\alpha_{1,qD}(0)+\delta_1\right]^2 \frac{\sigma_a^2\sigma_v^2}{\sigma_a^2+\sigma_v^2} + (1-\delta_1)^2\alpha_{5,qD}(0)^2(\sigma_a^2+\sigma_v^2) \tag{122}$$

Taking the second-order component of the average capital market accumulation equation and the average asset market clearing condition yields the same expressions for $k_{t+1}^A(2)$ and $q_t^A(2)$ as before, minus the terms in h_t. In particular, this implies that $\alpha_{qA}(1) = 0$ and that the matrix consisting of the first 4 rows and columns of $A_{qA}(0)$ is the same as $A_{qA}(0)$ we derived before. The other elements of the now expanded matrix $A_{qA}(0)$ are all zero. Taking the second-order component of the difference in asset market clearing conditions yields the same expression for $z_t^A(2)$ as before, but it depends on $q^D(2)$ and $k_{t+1}^D(2)$ that we still need to solve.

The second-order component of the average portfolio Euler equation still implies $[E_t er_{t+1}](2) = 0$, which now becomes

$$\begin{aligned} 0 &= lin_t(2) + quad_t(2) + \mu_{er,1}(1)\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^D + \mu_{er,2}(1)\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^A \\ &\quad +\mu_{er,6}(0)\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 v_t^D v_t^A + \mu_{er,9}(0)(\sigma_a^2+\sigma_v^2) + \frac{1}{4}\mu_{er,10}(0)(\sigma_a^2+\sigma_v^2) \end{aligned}$$

Here we used that $\mu_{er,4}(0) = \mu_{er,5}(0) = 0$, which follows from $A_{11,qD}(0) = A_{22,qD}(0) = 0$. This was the case under private information and we find below that the first 4 rows and columns of $A_{qD}(0)$ remain the same as under private information.

We have

$$\begin{aligned} lin_t(2) &= \left[(1-\delta_1)\alpha_{qD}(0) + \delta_1\left(I_1 - \omega I_3\right)\right]\bar{S}_{t+1}(2) + (1-\delta_1)\alpha_{qD}(1)\bar{S}_{t+1}(1) \\ &\quad -\alpha_{qD}(0)S_t(2) - \alpha_{qD}(1)S_t(1) \\ quad_t(2) &= (1-\delta_1)\bar{S}_{t+1}(1)' A_{qD}(0)\bar{S}_{t+1}(1) - S_t(1)' A_{qD}(0) S_t(1) \\ &\quad +\delta_2 \bar{S}_{t+1}(1)'\left(I_1 - \omega I_3 - \alpha_{qD}(0)\right)'\left(I_2 - \omega I_4 - \alpha_{qA}(0)\right)\bar{S}_{t+1}(1) \end{aligned}$$

Recall that $k_{t+1}^A(1) = \left[\frac{1}{1+\xi}I_2 + \frac{1+\xi-\omega}{1+\xi}I_4\right] S_t(1) = \left[\frac{1}{1+\xi}I_2 + \frac{1+\xi-\omega}{1+\xi}I_4\right] S_{1:4,t}(1)$ and $k_{t+1}^D(1) = \left[\frac{1}{\xi}\alpha_{qD}(0) + I_3\right] S_t(1) = \left[\frac{1}{\xi}\alpha_{1:4,qD}(0) + I_3\right] S_{1:4,t}(1) + \frac{1}{\xi}\alpha_{5,qD}(0)v_t^D$.

We write $\bar{S}_{t+1}(1)$ as follows:

$$\bar{S}_{t+1}(1) = \begin{vmatrix} \rho a_t^D \\ \rho a_t^A \\ k_{t+1}^D(1) \\ k_{t+1}^A(1) \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} \rho I_1 \\ \rho I_2 \\ \frac{1}{\xi}\alpha_{1:4,qD}(0) + I_3 \\ \frac{1}{1+\xi} I_2 + \frac{1+\xi-\omega}{1+\xi} I_4 \\ 0 \\ 0 \end{vmatrix} S_{1:4,t}(1) + \begin{vmatrix} 0 \\ 0 \\ \frac{1}{\xi}\alpha_{5,qD}(0) \\ 0 \\ 0 \\ 0 \end{vmatrix} v_t^D$$

The last two elements of $\bar{S}_{t+1}(1)$ are zero and:

$$\bar{S}_{1:4,t+1}(1) = N_1(0) S_{1:4,t}(1) + N_3(0) v_t^D$$

where $N_1(0)$ and $N_3(0)$ are as before.

With this, we write:

$$\begin{aligned}
lin_t(2) &= [(1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1] k_{t+1}^D(2) - \alpha_{3,qD}(0) k_t^D(2) \\
&\quad + \alpha_{1:4,qD}(1)\left[(1-\delta_1)N_1(0) - I_{4x4}\right] S_{1:4,t}(1) \\
&\quad + \left[(1-\delta_1)\alpha_{1:4,qD}(1) N_3(0) - \alpha_{5,qD}(1)\right] v_t^D - \alpha_{6,qD}(1) v_t^A \\
quad_t(2) &= S_{1:4,t}(1)' \left[(1-\delta_1)\left[N_1(0)\right]' \left[A_{4x4,qD}(0) + H(0)\right] N_1(0) - A_{4x4,qD}(0)\right] S_{1:4,t}(1) \\
&\quad - 2S_{5:6,t}(1)' A_{5:6,1:4,qD}(0) S_{1:4,t}(1) - S_{5:6,t}(1)' A_{5:6,5:6,qD}(0) S_{5:6,t}(1) \\
&\quad + 2(1-\delta_1)\left[N_3(0)\right]' \left[A_{4x4,qD}(0) + H(0)\right] N_1(0) S_{1:4,t}(1) v_t^D \\
&\quad + (1-\delta_1)\left[N_3(0)\right]' \left[A_{4x4,qD}(0) + H(0)\right] N_3(0) \left(v_t^D\right)^2
\end{aligned}$$

$k_{t+1}^D(2)$, which enters the expression for $lin_t(2)$, is determined by the second-order component of the difference in capital accumulation equations, which gives (using $\alpha_{5,qA}(0) = \alpha_{6,qA}(0) = \alpha_{6,qD}(0) = 0$):

$$\begin{aligned}
k_{t+1}^D(2) &= k_t^D(2) + \frac{1}{\xi} q_t^D(2) + \frac{\xi - 1}{\xi^2} q_t^A(1) q_t^D(1) \\
&= \left(I_3 + \frac{1}{\xi}\alpha_{1:4,qD}(0)\right) S_{1:4,t}(2) + \frac{\xi - 1}{\xi^2} S_{1:4,t}(1)' \alpha_{1:4,qA}(0)' \alpha_{1:4,qD}(0) S_{1:4,t}(1) \\
&\quad + \frac{\xi - 1}{\xi^2} \alpha_{5,qD}(0) \alpha_{1:4,qA}(0) S_{1:4,t}(1) v_t^D \\
&\quad + \frac{1}{\xi}\alpha_{1:4,qD}(1) S_{1:4,t}(1) + \frac{1}{\xi}\alpha_{5,qD}(1) v_t^D + \frac{1}{\xi}\alpha_{6,qD}(1) v_t^A \\
&\quad + \frac{1}{\xi} S_{1:4,t}(1)' A_{4x4,qD}(0) S_{1:4,t}(1) + \frac{1}{\xi} 2S_{5:6,t}(1)' A_{5:6,1:4,qD}(0) S_{1:4,t}(1) \\
&\quad + \frac{1}{\xi} S_{5:6,t}(1)' A_{5:6,5:6,qD}(0) S_{5:6,t}(1)
\end{aligned}$$

We can then substitute these results in the condition $E_t(er_{t+1})(2) = 0$. It is immediate that $A_{1:4,1:4,qD}(0)$ is the same as before and $\alpha_{1:4,qD}(1) = 0$ as before. This leaves us with terms that involve v_t^D and v_t^A. In what follows it is useful to write $\eta = (1-\delta_1)\alpha_{3,qD}(0) - \omega\delta_1$ and:

$$\begin{aligned}
\mu_{er,1}(1) &= \begin{bmatrix} 2(1-\delta_1)A_{1,1:4,qD}(0) \\ +\delta_2(1-\alpha_{1,qD}(0))\left(I_2 - \omega I_4 - \alpha_{1:4,qA}(0)\right) \end{bmatrix} \left(N_1(0)S_{1:4,t}(1) + N_3(0)v_t^D\right) \\
&= \bar{\mu}_{1s}S_{1:4,t}(1) + \bar{\mu}_{1v}N_3(0)v_t^D \qquad (123) \\
\mu_{er,2}(1) &= \begin{bmatrix} 2(1-\delta_1)A_{2,1:4,qD}(0) \\ +\delta_2\left(1-\alpha_{2,qA}(0)\right)\left(I_1 - \omega I_3 - \alpha_{1:4,qD}(0)\right) \end{bmatrix} \left(N_1(0)S_{1:4,t}(1) + N_3(0)v_t^D\right) \\
&= \bar{\mu}_{2s}S_{1:4,t}(1) + \bar{\mu}_{2v}N_3(0)v_t^D \qquad (124)
\end{aligned}$$

Note that $\bar{\mu}_{1v}N_3(0) = 0$ because the (1,3) elements of $A_{qD}(0)$ and $\alpha_{qA}(0)$ are zero. The condition $[E_t er_{t+1}](2) = 0$ is then written as:

$$\begin{aligned}
0 \;=\; & \left(\frac{\eta}{\xi}-1\right)\alpha_{6,qD}(1)v_t^A + \left(\frac{\eta}{\xi}-1\right)\alpha_{5,qD}(1)v_t^D \qquad (125) \\
& + \begin{bmatrix} \left(\frac{\eta}{\xi}-1\right)2A_{5,1:4,qD}(0) + \eta\frac{\xi-1}{\xi^2}\alpha_{5,qD}(0)\alpha_{1:4,qA}(0) \\ +2(1-\delta_1)\left[N_3(0)\right]'\left[A_{4x4,qD}(0) + H(0)\right]N_1(0) + \bar{\mu}_{1s}\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2} \end{bmatrix} S_{1:4,t}(1)\,v_t^D \\
& + \left[\left(\frac{\eta}{\xi}-1\right)2A_{6,1:4,qD}(0) + \bar{\mu}_{2s}\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right]S_{1:4,t}(1)\,v_t^A + \left(\frac{\eta}{\xi}-1\right)A_{66,qD}(0)\left(v_t^A\right)^2 \\
& + \left[(1-\delta_1)\left[N_3(0)\right]'\left[A_{4x4,qD}(0) + H(0)\right]N_3(0) + \left(\frac{\eta}{\xi}-1\right)A_{55,qD}(0)\right]\left(v_t^D\right)^2 \\
& + \left[\left(\frac{\eta}{\xi}-1\right)2A_{56,qD}(0) + \mu_{er,6}(0)\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 + \bar{\mu}_{2v}N_3(0)\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right]v_t^D v_t^A \\
& + \mu_{er,9}(0)(\sigma_a^2+\sigma_v^2) + \frac{1}{4}\mu_{er,10}(0)(\sigma_a^2+\sigma_v^2)
\end{aligned}$$

The first two terms imply $\alpha_{6,qD}(1) = \alpha_{5,qD}(1) = 0$.The coefficient on $\left(v_t^A\right)^2$ implies $A_{66,qD}(0) = 0$, which in turn implies that $\mu_{er,10}(0) = 0$. The first term in the bracket multiplying $(v_t^D)^2$ is zero because the (3,3) elements of $A_{qD}(0)$ and $H(0)$ are zero. Setting the coefficient on $(v_t^D)^2$ equal to zero then implies that $A_{55,qD}(0) = 0$, which in turn implies that $\mu_{er,9}(0) = 0$. The two constant terms at the end of the expression are therefore both zero.

Using our results for $\alpha_{5,qD}(0)$, $\alpha_{1:4,qA}(0)$ and $A_{4x4,qD}(0)$, setting the coefficient of $S_{1:4,t}(1) v_t^D$ equal to zero gives $A_{5,1:4,qD}(0)$:

$$A_{5,1:4,qD}(0) = \frac{1}{2}\frac{1}{1-\frac{\eta}{\xi}}\left\{\begin{array}{c} \eta\frac{\xi-1}{\xi^2}\alpha_{5,qD}(0)\alpha_{1:4,qA}(0) + \bar{\mu}_{1s}\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2} \\ +2(1-\delta_1)\left[N_3(0)\right]'\left[A_{4x4,qD}(0) + H(0)\right] N_1(0) \end{array}\right\} \tag{126}$$

The coefficient on $S_{1:4,t}(1) v_t^A$ gives $A_{6,1:4,qD}(0)$:

$$A_{6,1:4,qD}(0) = \frac{1}{2}\frac{1}{1-\frac{\eta}{\xi}}\bar{\mu}_{2s}\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2} \tag{127}$$

As we know $\mu_{er,6}(0) = 2(1-\delta_1)A_{12,qD}(0) + \delta_2(1-\alpha_{1,qD}(0))(1-\alpha_{2,qA}(0))$ the coefficient on $v_t^D v_t^A$ gives $A_{56,qD}(0)$:

$$A_{56,qD}(0) = \frac{1}{2}\frac{1}{1-\frac{\eta}{\xi}}\left[\mu_{er,6}(0)\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 + \bar{\mu}_{2v}N_3(0)\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right] \tag{128}$$

We finally need to compute $z_t^D(1)$, which follows from the third-order component of the difference in portfolio Euler equations:

$$z_t^D(1) = -z^D(0)\frac{[E_t er_{t+1}^2](3)}{[var_t(er_{t+1})](2)} \tag{129}$$

We have

$$E_t er_{t+1}^2(3) = 2E_t er_{t+1}(1) er_{t+1}(2) \tag{130}$$

Since the expectation of both $er_{t+1}(1)$ and $er_{t+1}(2)$ are zero, we only need their unexpected components.

Define $x_{t+1}^A = a_{t+1}^A - \omega k_{t+1}^A - q_{t+1}^A$ and $x_{t+1}^D = a_{t+1}^D - \omega k_{t+1}^D - q_{t+1}^D$. Then we have:

$$\begin{aligned} x_{t+1}^A(1) &= (1-\alpha_{2,qA}(0))\rho a_t^A - (\omega + \alpha_{4,qA}(0))k_{t+1}^A(1) + (1-\alpha_{2,qA}(0))\varepsilon_{t+1}^A \\ x_{t+1}^D(1) &= (1-\alpha_{1,qD}(0))\rho a_t^D - (\omega + \alpha_{3,qD}(0))k_{t+1}^D(1) + (1-\alpha_{1,qD}(0))\varepsilon_{t+1}^D - \alpha_{5,qD}(0)v_{t+1}^D \end{aligned}$$

Using

$$er_{t+1} = (1-\delta_1)q_{t+1}^D - q_t^D + \delta_1(a_{t+1}^D - \omega k_{t+1}^D) + \delta_2(a_{t+1}^A - \omega k_{t+1}^A - q_{t+1}^A)(a_{t+1}^D - \omega k_{t+1}^D - q_{t+1}^D) \tag{131}$$

the unexpected component of $er_{t+1}(1)$ is:

$$\mu_{er,1}(0)\varepsilon_{t+1}^D + (1-\delta_1)\alpha_{5,qD}(0)v_{t+1}^D \tag{132}$$

and the unexpected component of $er_{t+1}(2)$ is:

$$
\begin{aligned}
&\mu_{er,1}(1)\varepsilon_{t+1}^D + \mu_{er,2}(1)\varepsilon_{t+1}^A + \mu_{er,6}(0)\varepsilon_{t+1}^D\varepsilon_{t+1}^A \\
&+2(1-\delta_1)A_{15,qD}(0)\varepsilon_{t+1}^D v_{t+1}^D + 2(1-\delta_1)A_{25,qD}(0)\varepsilon_{t+1}^A v_{t+1}^D \\
&+2(1-\delta_1)A_{16,qD}(0)\varepsilon_{t+1}^D v_{t+1}^A + 2(1-\delta_1)A_{26,qD}(0)\varepsilon_{t+1}^A v_{t+1}^A + 2(1-\delta_1)A_{56,qD}(0)v_{t+1}^D v_{t+1}^A \\
&+2(1-\delta_1)v_{t+1}^D A_{5,1:4qD}N_3(0)v_t^D + 2(1-\delta_1)v_{t+1}^A A_{6,1:4qD}N_3(0)v_t^D \\
&+2(1-\delta_1)v_{t+1}^D A_{5,1:4,qD}(0)N_1(0)S_t(1) + 2(1-\delta_1)v_{t+1}^A A_{6,1:4,qD}(0)N_1(0)S_t(1) \\
&-\delta_2\alpha_{5,qD}(0)v_{t+1}^D\left((1-\alpha_{2,qA}(0))\rho a_t^A - (\omega+\alpha_{4,qA}(0))k_{t+1}^A(1) + (1-\alpha_{2,qA}(0))\varepsilon_{t+1}^A\right)
\end{aligned}
\tag{133}
$$

We now turn to $E_t er_{t+1}(1)er_{t+1}(2)$. An important point is that v_{t+1}^D and v_{t+1}^A are of expected value zero and independent from the ε_{t+1}. Also use again that $E_t v_{t+1}^D v_{t+1}^A = 0$. We then get:

$$
\begin{aligned}
&E_t er_{t+1}(1)er_{t+1}(2) = \\
&\mu_{er,1}(0)\left[\mu_{er,1}(1)E_t\left(\varepsilon_{t+1}^D\right)^2 + \mu_{er,2}(1)E_t\varepsilon_{t+1}^A\varepsilon_{t+1}^D + \mu_{er,6}(0)E_t\left(\varepsilon_{t+1}^D\right)^2\varepsilon_{t+1}^A\right] \\
&+(1-\delta_1)\alpha_{5,qD}(0)\left[\begin{array}{c}
2(1-\delta_1)A_{15,qD}(0)E_t\varepsilon_{t+1}^D\left(v_{t+1}^D\right)^2 + 2(1-\delta_1)A_{25,qD}(0)E_t\varepsilon_{t+1}^A\left(v_{t+1}^D\right)^2 \\
+2(1-\delta_1)A_{56,qD}(0)E_t\left(v_{t+1}^D\right)^2 v_{t+1}^A + 2(1-\delta_1)E_t\left(v_{t+1}^D\right)^2 A_{5,1:4qD}N_3(0)v_t^D \\
+2(1-\delta_1)E_t\left(v_{t+1}^D\right)^2 A_{5,1:4,qD}(0)N_1(0)S_t(1) \\
-\delta_2\alpha_{5,qD}(0)E_t\left(v_{t+1}^D\right)^2\left(\begin{array}{c}(1-\alpha_{2,qA}(0))\rho a_t^A - (\omega+\alpha_{4,qA}(0))k_{t+1}^A(1) \\ +(1-\alpha_{2,qA}(0))\varepsilon_{t+1}^A\end{array}\right)
\end{array}\right]
\end{aligned}
$$

We can write the following moments:

$$
\begin{aligned}
&E_t\left(v_{t+1}^D\right)^2 = 2\left(\sigma_a^2+\sigma_v^2\right) \\
&E_t\left(v_{t+1}^D\right)^2 v_{t+1}^A = E_t\left((v_{t+1}^H)^3 + (v_{t+1}^F)^3 - (v_{t+1}^H)^2 v_{t+1}^F - (v_{t+1}^F)^2 v_{t+1}^H\right) = 0
\end{aligned}
$$

We thus get:

$$
\begin{aligned}
&E_t er_{t+1}(1)er_{t+1}(2) = \mu_{er,1}(0)\mu_{er,6}(0)E_t\left(\varepsilon_{t+1}^D\right)^2\varepsilon_{t+1}^A \\
&+\mu_{er,1}(0)\left[\mu_{er,1}(1)\left[\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2\left(v_t^D\right)^2 + 2\frac{\sigma_a^2\sigma_v^2}{\sigma_a^2+\sigma_v^2}\right] + \mu_{er,2}(1)\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2 v_t^D v_t^A\right] \\
&+(1-\delta_1)\alpha_{5,qD}(0)\left[\begin{array}{c}
4(1-\delta_1)\sigma_a^2\left[A_{15,qD}(0)v_t^D + A_{25,qD}(0)v_t^A\right] \\
+4(1-\delta_1)\left(\sigma_a^2+\sigma_v^2\right)A_{5,1:4qD}\left[N_1(0)S_t(1) + N_3(0)v_t^D\right] \\
-\delta_2\alpha_{5,qD}(0)2\left(\sigma_a^2+\sigma_v^2\right)\left(\begin{array}{c}(1-\alpha_{2,qA}(0))\rho a_t^A - (\omega+\alpha_{4,qA}(0))k_{t+1}^A(1) \\ +(1-\alpha_{2,qA}(0))\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^A\end{array}\right)
\end{array}\right]
\end{aligned}
$$

The final element is to compute $E_t\left(\varepsilon_{t+1}^D\right)^2\varepsilon_{t+1}^A$:

$$E_t\left(\varepsilon_{t+1}^D\right)^2\varepsilon_{t+1}^A=\frac{1}{2}E_t\left[\begin{array}{c}\left(\varepsilon_{H,t+1}\right)^3-\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\\ +\left(\varepsilon_{F,t+1}\right)^3-\varepsilon_{H,t+1}\left(\varepsilon_{F,t+1}\right)^2\end{array}\right]$$

We have:

$$\begin{aligned}
\left[E_t\left(\varepsilon_{H,t+1}\right)^3\right](3) &= \left[\left[E_t\left(\varepsilon_{H,t+1}\right)\right](1)\right]^3+3\left[E_t\left(\varepsilon_{H,t+1}\right)\right](1)\left[Var\left(\varepsilon_{H,t+1}\right)\right](2)\\
&= \left[\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^H\right]^3+3\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2\sigma_v^2v_t^H\\
\left[E_t\left(\varepsilon_{F,t+1}\right)^3\right](3) &= \left[\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^F\right]^3+3\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2\sigma_v^2v_t^F\\
\left[E_t\left(\varepsilon_{H,t+1}\right)^2\varepsilon_{F,t+1}\right](3) &= \left[E_t\left(\varepsilon_{F,t+1}\right)\right](1)\left[E_t\left(\varepsilon_{H,t+1}^2\right)\right](2)\\
&= \left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^3\left(v_t^H\right)^2v_t^F+\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2\sigma_v^2v_t^F\\
\left[E_t\left(\varepsilon_{F,t+1}\right)^2\varepsilon_{H,t+1}\right](3) &= \left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^3\left(v_t^F\right)^2v_t^H+\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2\sigma_v^2v_t^H
\end{aligned}$$

Thus:

$$E_t\left(\varepsilon_{t+1}^D\right)^2\varepsilon_{t+1}^A=\frac{1}{2}\left[\begin{array}{c}\left[\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^H\right]^3+\left[\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}v_t^F\right]^3+2\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2\sigma_v^2v_t^H+2\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^2\sigma_v^2v_t^F\\ -\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^3\left(v_t^H\right)^2v_t^F-\left(\frac{\sigma_a^2}{\sigma_a^2+\sigma_v^2}\right)^3\left(v_t^F\right)^2v_t^H\end{array}\right]$$