MODEL THEORY OF OPERATOR ALGEBRAS II: MODEL THEORY

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Abstract. We introduce a version of logic for metric structures suitable for applications to C*-algebras and tracial von Neumann algebras. We also prove a purely model-theoretic result to the effect that the theory of a separable metric structure is stable if and only if all of its ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic even when the Continuum Hypothesis fails.

1. Introduction

The present paper is a companion to [10]. The latter paper was written in a way that completely suppressed explicit use of logic, and model theory in particular, in order to be more accessible to operator algebraists. Among other results, we will prove a metatheorem (Theorem 5.6) that explains results of [10], as well as the word 'stability' in its title.

We will study operator algebras using a slightly modified version of the *model theory for metric structures*. This is a logical framework whose semantics are well-suited for the approximative conditions of analysis; as a consequence it plays the same role for analytic ultrapowers as first order model theory plays for classical (set theoretic) ultrapowers. We show that the continuum hypothesis (CH) implies that all ultrapowers of a separable metric structure are isomorphic, but under the negation of CH this happens if and only if its theory is *stable*. Stability is defined in logical terms (the space of φ-types over a separable model is itself separable with a suitable topology), but it can be characterized as follows: a theory is *not* stable if and only if one can define arbitrarily long finite "uniformly well-separated" totally ordered sets in any model, a condition called the *order property*. Provided that the class of models under consideration (e.g., II_1 factors) is defined by a theory – not always obvious or even true – this brings the main question back into the arena of operator algebras. To deduce the existence of nonisomorphic ultrapowers under the negation of CH, one needs to establish the order property by defining appropriate posets. We proved in [10] that all infinite-dimensional C*-algebras and II_1 factors have the order property, while tracial von Neumann algebras of type I do not. In a sequel paper we will use the logic developed here to obtain new results about isomorphisms and embeddings between II_1 factors and their ultrapowers.

We now review some facts and terminology for operator algebraic ultrapowers that we will use throughout the paper; this is reproduced for convenience from [10].

A von Neumann algebra M is *tracial* if it is equipped with a faithful normal tracial state tr. A finite factor has a unique tracial state which is automatically normal. The metric induced by the ℓ^2-norm, $\|a\|_2 = \sqrt{\mathrm{tr}(a^*a)}$, is not complete on M, but it is complete on the (operator

Date: March 23, 2013.
The first two authors are partially supported by NSERC.

norm) unit ball of M. The completion of M with respect to this metric is isomorphic to a Hilbert space (see, e.g., [4] or [17]).

The algebra of all sequences in M bounded in the operator norm is denoted by $\ell^\infty(M)$. If $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then

$$c_{\mathcal{U}} = \{\vec{a} \in \ell^\infty(M) : \lim_{i\to\mathcal{U}} \|a_i\|_2 = 0\}$$

is a norm-closed two-sided ideal in $\ell^\infty(M)$, and the *tracial ultrapower* $M^{\mathcal{U}}$ (also denoted by $\prod_{\mathcal{U}} M$) is defined to be the quotient $\ell^\infty(M)/c_{\mathcal{U}}$. It is well-known that $M^{\mathcal{U}}$ is tracial, and a factor if and only if M is—see, e.g., [4] or [24]; this also follows from axiomatizability (§3.2) and Łoś's theorem (Proposition 4.3 and the remark afterwards).

Elements of $M^{\mathcal{U}}$ will either be denoted by boldface Roman letters such as $\mathbf{a}$ or represented by sequences in $\ell^\infty(M)$. Identifying a tracial von Neumann algebra M with its diagonal image in $M^{\mathcal{U}}$, we will also work with the *relative commutant* of M in its ultrapower,

$$M' \cap M^{\mathcal{U}} = \{\mathbf{b} : (\forall a \in M) a\mathbf{b} = \mathbf{b}a\}.$$

Tracial ultrapowers were first constructed in the 1950s and became standard tools after the groundbreaking papers of McDuff ([19]) and Connes ([7]). Roughly speaking, the properties of an ultrapower are the approximate properties of the initial object; see [23] for a recent discussion of this.

In defining ultrapowers for C*-algebras (resp. groups with bi-invariant metric), $c_{\mathcal{U}}$ is taken to be the sequences that converge to zero in the operator norm (resp. converge to the identity in the metric ([21])). All these constructions are special cases of the ultrapower/ultraproduct of metric structures (see §2, also [1] or [13]).

2. Logic

The purpose of this section is to introduce a logic which has some features geared to the treatment of C^*-algebras and von Neumann algebras. In a treatment of such structures in bounded continuous logic (see [3]), it is typical to consider different sorts of balls of increasing radius. The logic presented here is entirely equivalent to that formulation but allows us to introduce function symbols like $+$ and $\cdot$ without treating them as infinitely many different functions mapping between sorts. This distinction is somewhat cosmetic but the treatment of terms in this logic highlights an issue that is common to both this logic and the multi-sorted version. Details are given below but to make clear what is at stake, suppose we are considering a normed linear space and we wish to assert that the unit ball is convex. The operation $+$ when restricted to the unit ball would most naturally map to the ball of radius 2. Scalar multiplication by $1/2$ maps the ball of radius 2 into the unit ball and so a natural way to set things up would be to have the term $(x + y)/2$ send the unit ball to itself and so the syntax guarantees that the unit ball is convex. If on the other hand, the scalar $1/2$ on the ball of radius 2 was said to have range that same ball (a logical possibility), then $(x + y)/2$ syntactically would only map the unit ball to the ball of radius 2 and we would need to have an axiom that said that this term in fact has range in the unit ball. Issues of the axiomatizability of the classes of structures we are dealing with are bound up with the choice of range of terms in our language and are highlighted below.

2.1. **Language.** A language consists of

- Sorts, $\mathcal{S}$, and for each sort $S \in \mathcal{S}$, a set of domains $\mathcal{D}_S$ meant to be domains of quantification, and a privileged relation symbol d_S intended to be a metric. Each sort comes with a distinct set of variables.
- Sorted functions, $f : S_1 \times \ldots \times S_n \to S$ together with, for every choice of domains $D_i \in \mathcal{D}_{S_i}$, a $D^f_{\bar{D}} \in \mathcal{D}_S$ and for each i, a uniform continuity modulus $\delta_i^{\bar{D},\tau}$, i.e., a real-valued function on $\mathbb{R}$, where $\bar{D} = \langle D_1, \ldots, D_n\rangle$.
- Sorted relations R on $S_1 \times \ldots \times S_n$ such that for every choice of domains $\bar{D}$ as above, there is a number $N^R_{\bar{D}}$ as well as uniform continuity moduli dependent on i and $\bar{D}$.
- Terms are formed by the usual composition of function symbols and variables. They inherit codomains and series of uniform continuity moduli from this composition.

2.2. **Structures.** A structure $\mathcal{M}$ assigns to each sort $S \in \mathcal{S}$, $M(S)$, a metric space where d_S is interpreted as the metric. For each $D \in \mathcal{D}_S$, $M(D)$ is a subset of $M(S)$ complete with respect to d_S. The collection $\{M(D) : D \in \mathcal{D}_S\}$ covers $M(S)$.

Terms τ are interpreted as functions on a structure in the usual manner. If τ^M is the interpretation of τ and $\bar{D}$ is a choice of domains from the relevant sorts then $\tau^M : M(\bar{D}) \to M(D^\tau_{\bar{D}})$ and τ^M is uniformly continuous as specified by the $\delta^{\bar{D},\tau}$'s when restricted to $M(\bar{D})$. This means for instance that for every $\epsilon > 0$, if $a, b \in M(D_1)$ and $c_i \in M(D_i)$ for $i = 2, \ldots, n$ then $d_S(a, b) < \delta_1^{\bar{D},\tau}(\epsilon)$ implies $d_{S'}(\tau(a, \bar{c}), \tau(b, \bar{c})) \le \epsilon$, where S is the sort associated to D_1 and S' is the sort associated with the range of τ.

Sorted relations are maps $R^M : S_1 \times \ldots \times S_n \to \mathbb{R}$. They are handled similarly to sorted functions; uniform continuity is as above when restricted to the appropriate domains and a relation R is bounded in absolute value by $N^R_{\bar{D}}$ when restricted to $M(\bar{D})$.

2.3. **Examples.**

2.3.1. *C*-algebras.* We will think of a C*-algebra A as a one-sorted structure with sort U for the algebra itself. The domains for U are D_n for every $n \in \mathbb{N}$ and are interpreted as all $x \in A$ with $\|x\| \le n$. The metric on U is

$$d_U(a, b) = \|a - b\|.$$

The functions in the language will be:

- The constant 0 which will be in D_1. Note it is a requirement of the language to say this.
- For every $\lambda \in \mathbb{C}$ a unary function symbol also denoted λ to be interpreted as scalar multiplication. For simplicity we shall write λx instead of $\lambda(x)$.
- A unary function symbol $*$ for involution.
- Binary function symbols $+$ and $\cdot$.

Prescribing the uniform continuity moduli is straightforward.

If A is a C*-algebra then there is a model, $\mathcal{M}(A)$, in $\mathcal{L}_{C^*}$ associated to it which is essentially A itself endowed with the domains D_n interpreted as the operator norm n-ball.

2.3.2. *Tracial von Neumann algebras.* Tracial von Neumann algebras will be treated as a one-sorted structure with domains D_n which as in the example of C*-algebras will be interpreted as the operator norm n-ball. The metric d will be the metric arising from the ℓ^2 norm coming from the trace.

The functions in the language are, in addition to functions from §2.3.1,

- The constant 1 in D_1.
- Two unary relation symbols tr^r and tr^i for the real and imaginary parts of the trace function. We will often just write tr and assume that the expression can be decomposed into the real and imaginary parts.

Again, this describes a language $\mathcal{L}_{\mathrm{Tr}}$ once we add the requirements about bounds on the range and uniform continuity.

If N is a tracial von Neumann algebra then there is a model, $\mathcal{M}(N)$, in $\mathcal{L}_{\mathrm{Tr}}$ associated to it which is essentially N itself with the domains interpreted as above.

Remark 2.1. We emphasize that the operator norm is not a part of the language, and that it is not even a definable relation. Note that all relations are required to be uniformly continuous functions, and $\|\cdot\|$ is not uniformly continuous with respect to $\|\cdot\|_2$.

2.3.3. *Unitary groups.* The syntax for logic of unitary groups is simpler than that of tracial von Neumann algebras or C*-algebras. In this case the metric is bounded and therefore we can have one domain are equal to the universe U. We have function symbols for the identity, inverse and the group operation. Since in this case our logic reduces to the standard logic of metric structures as introduced in [1] we omit the straightforward details and continue this practice of suppressing the details for unitary groups throughout this section.

2.4. **Syntax.**

- Formulas:
  - If R is a relation and $\tau_1, \dots, \tau_n$ are terms then $R(\tau_1, \dots, \tau_n)$ is a basic formula.
  - If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $\varphi_1, \dots, \varphi_n$ are formulas, then $f(\varphi_1, \dots, \varphi_n)$ is a formula.
  - If $D \in \mathcal{D}_S$ and φ is a formula then both $\sup_{x \in D} \varphi$ and $\inf_{x \in D} \varphi$ are formulas.
- Formulas are interpreted in the obvious manner in structures. The boundedness of relations when restricted to domains is essential to guarantee that the sups and infs exist when interpreted. For a fixed formula φ and real number r, the expressions $\varphi \le r$ and $r \le \varphi$ are called *conditions* and are either true or false in a given interpretation in a structure.

2.4.1. *The expanded language.* In the above definition it was taken for granted that we have an infinite supply of distinct variables appearing in terms. In §4.3 below we shall need to introduce a set of new constant symbols $\mathbf{C}$. Each $c \in \mathbf{C}$ is assigned a sort $S(c)$ and a domain. In the *expanded language* $\mathcal{L}_{\mathbf{C}}$ both variables and constant symbols from $\mathbf{C}$ appear in terms. Formulas and sentences in $\mathcal{L}_{\mathbf{C}}$ are defined as above. Note that, since the elements of $\mathbf{C}$ are not variables, we do not allow quantification over them.

2.5. **Theories and elementary equivalence.** A sentence is a formula with no free variables. If φ is a sentence and $\mathcal{M}$ is a structure then the result of interpreting φ in $\mathcal{M}$ is a real number, $\varphi^{\mathcal{M}}$. The function which assigns these numbers to sentences is the *theory of*

$\mathcal{M}$, denoted by $\mathrm{Th}(\mathcal{M})$. Because we allow all continuous functions as connectives, in particular the functions $|x-\lambda|$, the theory of a model $\mathcal{M}$ is uniquely determined by its zero-set, $\{\varphi : \varphi^{\mathcal{M}} = 0\}$. We shall therefore adopt the convention that a set of sentences T is a theory and say that $\mathcal{M}$ *is a model of* T, $\mathcal{M} \models T$, if $\varphi^{\mathcal{M}} = 0$ for all $\varphi \in T$.

The following is proved by induction on the complexity of the definition of ψ.

Lemma 2.2. *Suppose $\mathcal{M}$ is a model and $\psi(\bar{x})$ is a formula, possibly with parameters from M. For every choice of $\bar{D}$ sequence of domains consistent with the sorts of the variables, $\psi^{\mathcal{M}}$ is a uniformly continuous function on $M(\bar{D})$ into a compact subset of $\mathbb{R}$.*

If $\Theta\colon \mathcal{M} \to \mathcal{N}$ is an isomorphism then $\psi^{\mathcal{M}} = \psi^{\mathcal{N}} \circ \Theta$. □

Two models $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. A map $\Theta\colon \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* if for all formulas ψ with parameters in M, we have $\psi^{\mathcal{M}} = \psi^{\mathcal{N}} \circ \Theta$.

If $\mathcal{M}$ is a submodel of $\mathcal{N}$ and the identity map from $\mathcal{M}$ into $\mathcal{N}$ is elementary then we say that $\mathcal{M}$ is an *elementary submodel* of $\mathcal{N}$. It is not difficult to see that every elementary embedding is an isomorphism onto its image,[1] but not vice versa.

3. Axiomatizability

Definition 3.1. A category $\mathcal{C}$ is **axiomatizable** if there is a language $\mathcal{L}$ (as above), theory T in $\mathcal{L}$, and a collection of conditions Σ such that $\mathcal{C}$ is equivalent to the category of models of T with morphisms given by maps that preserve Σ.

The reason for being a little fussy about axiomatizability is that in the cases we wish to consider, the models have more (albeit artificial) 'structure' than the underlying algebra (cf. §2.3.1 and §2.3.2). The language of the model will contain operation symbols for all the algebra operations (such as $+, \cdot$ and $*$) and possibly some distinguished constant symbols (such as the unit) and predicates (e.g., a distinguished state on a C*-algebra). It will also contain domains that are not part of the algebra's structure.

In particular then, when we say that we have axiomatized a class of algebras $\mathcal{C}$, we will mean that there is a first order continuous theory T and specification of morphisms such that

- for any $A \in \mathcal{C}$, there is a model $M(A)$ of T determined up to isomorphism;
- for any model M of T there is $A \in \mathcal{C}$ such that M is isomorphic to $M(A)$;
- if $A, B \in \mathcal{C}$ then there is a bijection between $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(M(A), M(B))$.

Proving that a category is axiomatizable frequently involves somewhat tedious syntactical considerations. However, once this is proved we can apply a variety of model-theoretic tools to study this category. In particular, we can immediately conclude that the category is closed under taking ultraproducts—a nontrivial theorem in the case of tracial von Neumann algebras. From here it also follows that some natural categories of operator algebras are not axiomatizable (see Proposition 6.1).

[1] an isomorphism in the appropriate category; in case of operator algebras this is interpreted as '*-isomorphism'

3.1. **Axioms for C*-algebras.** We continue the discussion of model theory of C*-algebras started in §2.3.1. First we introduce two notational shortcuts. If one wants to write down axioms to express that $\tau = \sigma$ for terms τ and σ then one can write

$$\varphi_{\bar{D}} := \sup_{\bar{a} \in \bar{D}} d_U(\tau(\bar{a}), \sigma(\bar{a}))$$

where $\bar{D}$ ranges over all possible choices of domains. Note that this is typically an infinite set of axioms. Remember that for a model to satisfy $\varphi_{\bar{D}}$, this sentence would evaluate to 0 in that model. If this sentence evaluates to 0 for all choices of $\bar{D}$ then clearly $\tau = \sigma$ in that model.

If one wants to write down axioms to express that $\varphi \geq \psi$ for formulas φ and ψ then one can write

$$\sup_{\bar{a} \in \bar{D}} \max(0, (\psi(\bar{a}) - \varphi(\bar{a})))$$

where $\bar{D}$ ranges over all possible choices of domains. Again, we will get the required inequality if all these sentences evaluate to 0 in a model.

Using the above conventions, we are taking the universal closures of the following formulas, where $x, y, z, a, b,$ range over the algebra and λ, μ range over the complex numbers.

Here are some sentences that evaluate to zero in a C*-algebra A. The first item guarantees that we have a $\mathbb{C}$-vector space.

(1) $x + (y + z) = (x + y) + z$, $x + 0 = x$, $x + (-x) = 0$ (where $-x$ is the scalar -1 acting on x), $x + y = y + x$, $\lambda(\mu x) = (\lambda\mu)x$, $\lambda(x + y) = \lambda x + \lambda y$, $(\lambda + \mu)x = \lambda x + \mu x$.
(2) $1x = x$, $x(yz) = (xy)z$, $\lambda(xy) = (\lambda x)y = x(\lambda y)$, $x(y + z) = xy + xz$; now we have a $\mathbb{C}$-algebra.
(3) $(x^*)^* = x$, $(x + y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda} x^*$.
(4) $(xy)^* = y^* x^*$.
(5) $d_U(x, y) = d_U(x - y, 0)$; we will write $\|x\|$ for $d_U(x, 0)$.
(6) $\|xy\| \leq \|x\|\|y\|$.
(7) $\|\lambda x\| = |\lambda|\|x\|$.
(8) (C*-equality) $\|xx^*\| = \|x\|^2$.
(9) $\sup_{a \in D_1} \|a\| \leq 1$.

One issue here is that these axioms are too weak to guarantee that D_1 is the operator norm unit ball. To get around this we expand the language of C*-algebras to include a function symbol τ_p for every *-polynomial p in one variable. The symbol τ_p will have the same uniform continuity modulus as p. In order to determine the proper codomains, for every n, let m be the least integer greater than or equal to $\max\{\|p(a)\| : a \in M, M \in \mathcal{C} \text{ and } \|a\| \leq n\}$ where $\mathcal{C}$ is the class of C*-algebras. We will require $\tau_p : D_n \to D_m$ and we will add the universally quantified axioms

(10) $\tau_p(x) = p(x)$

for all polynomials p. This will force the polynomial p to behave well with respect to where its range lands. To see the effect of these axioms, we do a small calculation.

Suppose that $\mathcal{M}$ is a structure that satisfies axioms 1 through 9 above. Suppose $a \in M$, $\|a\| \leq 1$ and $a \in D_n(\mathcal{M})$. Define

$$t_n(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ \frac{1}{\sqrt{x}} & 1 < x \leq n \end{cases}$$

and consider $f(u) = ut_n(u^*u)$. If we want to compute the norm of $f(u)$ for $\|u\| \leq n$, we see that $\|f(u)\|^2 = \|t_n(u^*u)u^*ut_n(u^*u)\| = \|g(u^*u)\|$ where $g(x) = xt_n^2(x)$. Since

$$g(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 1 & 1 < x \leq n \end{cases}$$

we obtain that the norm of $f(u)$ is at most 1 when $\|u\| \leq n$.

Now fix polynomials $p_k(x)$ which tend to $t_n(x)$ from below on the interval $[0, n]$. By doing a calculation similar to the one above, the *-polynomial $q_k = up_k(u^*u)$ sends operators of norm $\leq n$ to operators of norm ≤ 1. This means that τ_{q_k} sends elements of D_n to elements of D_1 by the specification of our language for C*-algebras. Moreover, $ap_k(a^*a)$ tends to a as k tends to infinity. Since $D_1(\mathcal{M})$ is complete, we obtain that $a \in D_1(\mathcal{M})$.

Proposition 3.2. *The class of C*-algebras is axiomatizable by theory $\mathbf{T}_{C^*}$ consisting of axioms* (1)–(10)*.*

Proof. It is clear that for a C*-algebra A the model $\mathcal{M}(A)$ as defined in §2.3.1 satisfies $\mathbf{T}_{C^*}$. Conversely, if a model $\mathcal{M}$ of $\mathcal{L}_{C^*}$ satisfies $\mathbf{T}_{C^*}$ then the algebra $A_{\mathcal{M}}$ obtained from $\mathcal{M}$ by forgetting the domains is a C*-algebra by Gel'fand-Naimark.

To see that this provides an equivalence of categories, we only need to show that $M(A_{\mathcal{M}}) \cong \mathcal{M}$. To see this, we must show that the domains on $\mathcal{M}$ are determined by $A_{\mathcal{M}}$. Since multiplication by a scalar r provides a bijection between the operator norm unit ball and the ball of radius r, it suffices to show that the operator norm unit ball and those elements of $D_1(\mathcal{M})$ coincide. By axiom 9, we have that the latter is contained in the former. The other direction is just the calculation we did immediately before the Proposition. □

3.2. **Axioms for tracial von Neumann algebras.** We continue our discussion of model theory of tracial von Neumann algebras from §2.3.2. Axioms for tracial von Neumann algebras and II_1 factors appear in the context of bounded continuous logic in [2]; those axioms are restricted to axiomatizing the norm one unit ball. We feel in this context axiomatizing von Neumann algebras in the logic described in the previous section makes the axioms more natural. Here are some sentences that evaluate to zero in a tracial von Neumann algebra N:

(11) All axioms (1)–(5) plus $1x = x = x1$ for the constant 1 of N. In case of (5) we will write $\|x\|_2$ for $d_U(x, 0)$.
(12) $\mathrm{tr}(x + y) = \mathrm{tr}(x) + \mathrm{tr}(y)$
(13) $\mathrm{tr}(x^*) = \overline{\mathrm{tr}(x)}$, $\mathrm{tr}(\lambda x) = \lambda\,\mathrm{tr}(x)$, $\mathrm{tr}(xy) = \mathrm{tr}(yx)$, $\mathrm{tr}(1) = 1$,
(14) $\mathrm{tr}(x^*x) = \|x\|_2^2$.

Any model of these axioms will be a tracial *-algebra. The remaining axiom will guarantee that the relationship between the domains and the 2-norm is correct.

(15) For every $n, m \in \mathbb{N}$,

$$\sup_{a \in D_n} \sup_{x \in D_m} \max\{0, \|ax\|_2 - n\|x\|_2\}$$

In addition to these axioms, we also introduce terms τ_p for all unary *-polynomials p as discussed above for C*-algebras.

Proposition 3.3. *The class of tracial von Neumann algebras is axiomatizable by theory $\mathbf{T}_{\mathrm{Tr}}$ consisting of axioms* (10)–(15)*.*

Proof. It is clear that for a tracial von Neumann algebra N the model $\mathcal{M}(N)$ as defined in §2.3.2 satisfies $\mathbf{T}_{\mathrm{Tr}}$. Assume $\mathcal{M}$ satisfies $\mathbf{T}_{\mathrm{Tr}}$. To see that in the sort U we have a tracial von Neumann algebra suppose A is the underlying set for U in $\mathcal{M}$. Then A is a complex pre-Hilbert space with inner product given by $\mathrm{tr}(y^*x)$. Left multiplication by $a \in A$ is a linear operator on A and axiom (15) guarantees that a is bounded. The operation $*$ is the adjoint because for all x and y we have $\langle ax, y\rangle = \mathrm{tr}(y^*ax) = \mathrm{tr}((a^*y)^*x) = \langle x, a^*y\rangle$. Thus A is faithfully represented as a *-algebra of Hilbert space operators. We know that $D_n(A)$ is complete with respect to the 2-norm for all n and the 2-norm induces the strong operator topology on A in this representation; it follows from the Kaplansky density theorem that A is a tracial von Neumann algebra.

As in the case of C*-algebras above, to show that we have an equivalence of categories, it will suffice to show that if $\mathcal{M}$ is a model of the $\mathbf{T}_{\mathrm{Tr}}$ then $D_1(A)$ is given by the operator norm unit ball on A. Axiom (15) guarantees that $a \in D_1(A)$ then $\|a\| \leq 1$ and the functional calculus argument from the proof of Proposition 3.2 shows $D_1(A)$ equals the operator norm unit ball. □

For a in a tracial von Neumann algebras define the following:

$$\xi(a) = \sqrt{\|a\|_2^2 - \mathrm{tr}^2(a)},$$
$$\eta(a) = \sup_{b \in D_1} \|ab - ba\|_2.$$

Since ξ and η are interpretations of terms in the language of tracial von Neumann algebras, the following is a sentence of this language.

(16) $\sup_{a \in D_1} \max\{0, (\xi(a) - \eta(a))\}$.

Also consider the axiom

(17) $\inf_{a \in D_1}(\|aa^* - (aa^*)^2\|_2 + |\,\mathrm{tr}(aa^*) - 1/\pi|)$.

Proposition 3.4. (1) *The class of tracial von Neumann factors is axiomatizable by the theory consisting of axioms* (10)–(16).

(2) *The class of II_1 factors is axiomatizable by the theory $\mathbf{T}_{II_1}$ consisting of axioms* (10)–(17).

Proof. For (1), by Proposition 3.3, it suffices to prove that if M is a tracial von Neumann algebra then axiom (16) holds in M if and only if M is a factor. If it is not a factor, let p be a nontrivial central projection. Then $\xi(p) = \sqrt{\mathrm{tr}(p) - \mathrm{tr}(p)^2} > 0$ but $\eta(p) = 0$, therefore (16) fails in M. If it is a factor, the inequality $\eta(a) \geq \xi(a)$ follows from [10, Lemma 4.2].

For (2) we need to show that axiom (17) holds in a tracial factor M if and only if M is type II_1. When M is type II_1, (17) is satisfied by taking a to be a projection of trace $1/\pi$. On the other hand, a tracial factor M not of type II_1 is some matrix factor $\mathbb{M}_k$. If $\mathbb{M}_k$ were to satisfy (17), by compactness of the unit ball there would be $a \in \mathbb{M}_k$ satisfying $\|(aa^*) - (aa^*)^2\|_2 = 0$ and $|\,\mathrm{tr}(aa^*) - 1/\pi| = 0$. Thus $aa^* \in \mathbb{M}_k$ would be a projection of trace $1/\pi$, which is impossible. (Of course this argument still works if $1/\pi$ is replaced with any irrational number in $(0, 1)$.) □

4. Model-theoretic toolbox

In the present section we introduce variants of some of the standard model-theoretic tools for the logic described in §2.

4.1. **Ultraproducts.** Assume $\mathcal{M}_i$, for $i \in I$, are models of the same language and $\mathcal{U}$ is an ultrafilter on I. The *ultraproduct* $\prod_{\mathcal{U}} \mathcal{M}_i$ is a model of the same language defined as follows.

In a model $\mathcal{M}$, we write $S^{\mathcal{M}}$ and $D^{\mathcal{M}}$ for the interpretations S and D in $\mathcal{M}$. For each sort $S \in \mathcal{S}$, let

$$X_S = \{\bar{a} \in \prod_{i\in I} S^{\mathcal{M}_i} : \text{for some } D \in \mathcal{D}_S, \{i \in I : a_i \in D^{\mathcal{M}_i}\} \in \mathcal{U}\}.$$

For $\bar{a}$ and $\bar{b}$ in X_S, $d'_S(\bar{a}, \bar{b}) = \lim_{i\to\mathcal{U}} d_S^{\mathcal{M}_i}(a_i, b_i)$ defines a pseudo-metric on X_S. Let $S^{\mathcal{M}'}$ be the quotient space of X_S with respect to the equivalence $\bar{a} \sim \bar{b}$ iff $d'_S(\bar{a}, \bar{b}) = 0$ and let d_S be the associated metric. For $D \in \mathcal{D}_S$, let $D^{\mathcal{M}'}$ be the quotient of

$$\{\bar{a} \in X_S : \{i \in I : a_i \in D^{\mathcal{M}_i}\} \in \mathcal{U}\}.$$

All the functions and predicates are interpreted in the natural way. Their restrictions to each $\bar{D}$ are uniformly continuous and respect the corresponding uniform continuity moduli. If $\mathcal{M}_i = \mathcal{M}$ for all i then we call the ultraproduct an *ultrapower* and denote it by $\mathcal{M}^{\mathcal{U}}$. The 'generalized ultraproduct construction' as introduced in [13, p. 308–309] reduces to the model-theoretic ultraproduct in the case of both tracial von Neumann algebras and C*-algebras.

We record a straightforward consequence of the definitions and the axiomatizability, that the functors corresponding to taking the ultrapower and defining a model commute. The ultrapowers of C*-algebras and tracial von Neumann algebras are defined in the usual way.

Proposition 4.1. *If A is a C*-algebra or a tracial von Neumann algebra and $\mathcal{U}$ is an ultrafilter then $\mathcal{M}(A^{\mathcal{U}}) = \mathcal{M}(A)^{\mathcal{U}}$.*

Corollary 4.2. *A C*-algebra (or a tracial von Neumann algebra) A has nonisomorphic ultrapowers if and only if the model $\mathcal{M}(A)$ has nonisomorphic ultrapowers.*

Proof. This is immediate by Proposition 3.2, Proposition 3.3 and Proposition 4.1. □

It is worth remarking that although the proof of Proposition 4.1 is straightforward, this relies on a judicious choice of domains of quantification. In general, it is not true that if one defines domains for a metric structure then the domains have the intended or standard interpretation in the ultraproduct. Von Neumann algebras themselves are a case in point. If we had defined our domains so that D_n were those operators with l_2-norm less than or equal to n then there would be several problems. The most glaring is that these domains are not complete; even if one persevered to an ultraproduct, the resulting object would contain unbounded operators.

Ward Henson has pointed out to us that this same problem with domains manifests itself in pointed ultrametric spaces. If one defines domains as closed balls of radius n about the base point, there is no reason to expect that the domains in an ultraproduct will also be closed balls. This unwanted phenomenon can be avoided by imposing a geodesic-type condition on the underlying metric; see for instance [6, Section 1.8].

The following is Łoś's theorem, also known as the Fundamental Theorem of ultraproducts (see [1, Theorem 5.4]). It is proved by chasing the definitions.

Proposition 4.3. *Let $\mathcal{M}_i$, $i \in \mathbb{N}$, be a sequence of models of language $\mathcal{L}$, $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{M}_i$.*

(1) *If ϕ is an $\mathcal{L}$-sentence then $\phi^{\mathcal{N}} = \lim_{i\to\mathcal{U}} \phi^{\mathcal{M}_i}$.*

(2) *If ϕ is an $\mathcal{L}$-formula then $\phi^{\mathcal{N}}(\mathbf{a}) = \lim_{i\to\mathcal{U}} \phi^{\mathcal{M}_i}(a_i)$, where $(a_i\colon i \in \mathbb{N})$ is a representing sequence of $\mathbf{a}$.*
(3) *The diagonal embedding of a model $\mathcal{M}$ into $\mathcal{M}^{\mathcal{U}}$ is elementary.*

Together with the axiomatizability (Propositions 3.2 and 3.3) and Proposition 4.1, this implies the well-known fact that the ultraproduct of C*-algebras (tracial von Neumann algebras, II_1 factors, respectively) is a C*-algebra (tracial von Neumann algebra, II_1 factor, respectively).

In the setting of tracial von Neumann algebras, we have that for any formula $\phi(x_1, \dots, x_n)$ with variables from the algebra sort there is a uniform continuity modulus δ such that for every tracial von Neumann algebra $\mathcal{M}$, ϕ defines a function g on the operator norm unit ball of $\mathcal{M}$ which is uniformly continuous with respect to δ and naturally extends to the operator norm unit ball of any ultrapower of $\mathcal{M}$.

In [10] we dealt with functions g satisfying the properties in the previous paragraph and used them to define a linear ordering showing that some ultrapowers and relative commutants are nonisomorphic. Using model theory, we can interpret this in a more general context and instead of 'tracial von Neumann algebra' consider g defined with respect to any axiomatizable class of operator algebras. Clearly, Lemma 2.2 and Proposition 4.3 together imply the following, used in the proof of Theorem 5.6.

Corollary 4.4. *If ψ is an n-ary formula, then the function g defined to be the interpretation of ψ on a tracial von Neumann algebra M satisfies the following* [10, Properties 2.1]*:*

(G1) *g defines a uniformly continuous function on the n-th power of the unit ball of M; the uniform continuity does not depend on the particular algebra i.e. for every ϵ there is a δ independent of the choice of algebra;*
(G2) *For every ultrafilter $\mathcal{U}$ the function g can be canonically extended to the n-th power of the unit ball of the ultrapower $(M_{\leq 1})^{\mathcal{U}} = (M^{\mathcal{U}})_{\leq 1}$.* □

4.2. **Downwards Löwenheim–Skolem Theorem.** The cardinality of the language and the number of formulas are crude measures of the Löwenheim-Skolem cardinal for continuous logic. We define a topology on formulas relative to a given continuous theory in order to give a better measure.

Suppose T is a continuous theory in a language $\mathcal{L}$. Fix variables $\bar{x} = x_1 \dots x_n$ and domains $\bar{D} = D_1 \dots D_n$ consistent with the sorts of the x's. For formulas φ and ψ defined on $\bar{D}$, set

$$d^T_{\bar{D}}(\varphi(\bar{x}), \psi(\bar{x})) = \sup\{ \sup_{\bar{x}\in(\bar{D}^{\mathcal{M}})^n} |\varphi(\bar{x}) - \psi(\bar{x})| : \mathcal{M} \models T\}.$$

Now $d^T_{\bar{D}}$ is a pseudo-metric; let $\chi(T, \bar{D})$ be the density character of this pseudo-metric on the formulas in the variables $\bar{x}$ and define the density character of $\mathcal{L}$ with respect to T, $\chi(T)$, as $\sum_{\bar{D}} \chi(T, \bar{D})$.

We will say that $\mathcal{L}$ is *separable* if the density character of $\mathcal{L}$ is countable with respect to all $\mathcal{L}$-theories. Note that the languages considered in this paper, in particular $\mathcal{L}_{\mathrm{Tr}}$ and $\mathcal{L}_{C^*}$, are separable.

Proposition 4.5. *Assume $\mathcal{L}$ is a separable language. Then for every model $\mathcal{M}$ of $\mathcal{L}$ the set of all interpretations of formulas of $\mathcal{L}$ is separable in the uniform topology.*

Proof. Since we are allowing all continuous real functions as propositional connectives (§2.1) the set of formulas is not countable. However, a straightforward argument using polynomials with rational coefficients and the Stone–Weierstrass theorem gives a proof. □

The following is a version of the downward Löwenheim–Skolem theorem (cf. [1, Proposition 7.3]). Some of its instances have been rediscovered and applied in the context of C*-algebras (see, e.g., [22] or the discussion of SI properties in [4]). We use the notation $\chi(X)$ to represent the density character of a set X in some ambient topological space.

Theorem 4.6. *Suppose that $\mathcal{M}$ is a metric structure and $X \subseteq M$. Then there is $\mathcal{N} \prec \mathcal{M}$ such that $X \subseteq N$ and $\chi(\mathcal{N}) \leq \chi(\mathrm{Th}(\mathcal{M})) + \chi(X)$.*

Proof. Fix $\mathcal{F}$, a dense set of formulas, witnessing $\chi(\mathrm{Th}(\mathcal{M}))$. Define two increasing sequences $\langle X_n : n \in \mathbb{N}\rangle$ and $\langle E_n : n \in \mathbb{N}\rangle$ of subsets of M inductively so that:

(1) $X_0 = X$;
(2) E_n is dense in X_n and $\chi(X_n) = |E_n|$ for all $n \in \mathbb{N}$;
(3) $\chi(X_n) \leq \chi(\mathrm{Th}(\mathcal{M})) + \chi(X)$; and,
(4) for every rational number r, formula $\varphi(x, \bar{y}) \in \mathcal{F}$, domain D in the sort of the variable x and $\bar{a} = a_1, \dots, a_k \in E_n$ where k is the length of $\bar{y}$, if $\mathcal{M} \models \inf_{x \in D} \varphi(x, \bar{a}) \leq r$ then for every $n > 0$ there is $b \in X_{n+1} \cap D(\mathcal{M})$ such that $\mathcal{M} \models \varphi(b, \bar{a}) \leq r + (1/n)$.

It is routine to check that $\overline{\cup_{n\in\mathbb{N}} X_n}$ is the universe of an elementary submodel $\mathcal{N} \prec \mathcal{M}$ having the correct density character. □

Corollary 4.7. *Assume $\mathcal{L}$ is separable. If $\mathcal{M}$ is a model of $\mathcal{L}$ and X is an infinite subset of its universe, then $\mathcal{M}$ has an elementary submodel whose density character is not greater than that of X and whose universe contains X.*

4.3. **Types.** Suppose that $\mathcal{M}$ is a model in a language $\mathcal{L}$, $A \subseteq M$ and $\bar{x}$ is a tuple of free variables thought of as the type variables.

We follow [1, Remark 3.13] and say that a *condition* over A is an expression of the form $\varphi(\bar{x}, \bar{a}) \leq r$ where $\varphi \in \mathcal{L}$, $\bar{a} \in A$ and $r \in \mathbb{R}$. If $\mathcal{N} \succ \mathcal{M}$ and $\bar{b} \in N$ then $\bar{b}$ satisfies $\varphi(\bar{x}, \bar{a}) \leq r$ if $\mathcal{N}$ satisfies $\varphi(\bar{b}, \bar{a}) \leq r$.

Fix a tuple of domains $\bar{D}$ consistent with $\bar{x}$, i.e., if x_i is of sort S then D_i is a domain in S. A set of conditions over A is called a $\bar{D}$-type over A. A $\bar{D}$-type is *consistent* if for every finite $p_0 \subseteq p$ and $\epsilon > 0$ there is $\bar{b} \in \bar{D}(M)$ such that if "$\varphi(\bar{x}, \bar{a}) \leq r$" $\in p_0$ then M satisfies $\varphi(\bar{b}, \bar{a}) \leq r + \epsilon$. We say that a $\bar{D}$-type p over A is *realized* in $\mathcal{N} \succ \mathcal{M}$ if there is $\bar{a} \in \bar{D}(N)$ such that $\bar{a}$ satisfies every condition in p. The following proposition links these two notions:

Proposition 4.8. *The following are equivalent:*

(1) *p is consistent.*
(2) *p is realized in some $\mathcal{N} \succ \mathcal{M}$.*
(3) *p is realized in an ultrapower of $\mathcal{M}$.*

Proof. 3) implies 2) and 2) implies 1) are clear. To see that 1) implies 3), let $F \subseteq p \times \mathbb{R}_+$ be a finite set, and let $\bar{b}_F \in \bar{D}(M)$ satisfy $\varphi(\bar{x}, \bar{a}) \leq r + \epsilon$ for every $(\varphi(\bar{x}, \bar{a}) \leq r, \epsilon) \in F$. Let $\mathcal{U}$ be a non-principal ultrafilter over $\mathcal{P}_{fin}(p \times \mathbb{R}_+)$. Then p is realized by $(\bar{b}_F : F \in \mathcal{P}_{fin}(p \times \mathbb{R}_+))/\mathcal{U}$ in $\mathcal{M}^{\mathcal{U}}$. □

A maximal consistent $\bar{D}$-type is called *complete.* Let $S^{\bar{D}}(A)$ be the set of all complete $\bar{D}$-types over A. In fact, p is a complete $\bar{D}$-type over A iff p is the set of all conditions true for some $\bar{a} \in \bar{D}(\mathcal{N})$ where $\mathcal{N} \succ \mathcal{M}$, by Lemma 4.8 and Proposition 4.3.

Notation 4.9. Assume p is a complete type over A and $\phi(x, \bar{a})$ is a formula with parameters $\bar{a}$ in A. Since p is consistent and maximal, there is the unique real number $r = \sup\{s \in \mathbb{R}$: the condition $\phi(x, \bar{a}) \leq s$ is in $p\}$. In this situation we shall extend the notation by writing $\phi(p, \bar{a}) = r$. We shall also use expressions such as $|\phi(p, \bar{a}) - \phi(p, \bar{b})| > \varepsilon$.

We will also often omit the superscript $\bar{D}$ when it either does not matter or is implicit.
The set $S^{\bar{D}}(A)$ carries two topologies: the logic topology and the metric topology.
Fix φ, $\bar{a} \in A$ and $r \in \mathbb{R}$. A basic closed set in the logic topology has the form

$$\{p \in S^{\bar{D}}(A) : \varphi(\bar{x}, \bar{a}) \leq r \in p\}$$

The compactness theorem shows that this topology is compact and it is straightforward that it is Hausdorff.
We can also put a metric on $S^{\bar{D}}(A)$ as follows: for $p, q \in S^{\bar{D}}(A)$ define

$$d(p, q) = \inf\{d(a, b) : \text{there is an } \mathcal{N} \succ \mathcal{M}, a \text{ realizes } p \text{ and } b \text{ realizes } q\}.$$

The metric topology is in general finer than the logic topology due to the uniform continuity of formulas.

Example 4.10. Let $\mathcal{M}$ be a model corresponding to a tracial von Neumann algebra or a unital C*-algebra.

(1) The *relative commutant type* of $\mathcal{M}$ is the type over M consisting of all conditions of the form

$$d([a, x], 0) = 0$$

for $a \in M$.

(2) Another type over $\mathcal{M}$ consists of all conditions of the form

$$d(a, x) \geq \varepsilon$$

for $a \in M$ and a fixed $\varepsilon > 0$.

While the relative commutant type is trivially realized by the center of $\mathcal{M}$, the type described in (2) is never realized in $\mathcal{M}$. However, the second type is sometimes consistent. For instance, if $\mathcal{M}$ is an infinite dimensional C*-algebras then (2) is consistent. Hence not every consistent type over $\mathcal{M}$ is necessarily realized in $\mathcal{M}$.

4.4. **Saturation.** A model $\mathcal{M}$ of language $\mathcal{L}$ is *countably saturated* if for every countable subset X of the universe of $\mathcal{M}$, every consistent type over X is realized in $\mathcal{M}$. More generally, if κ is a cardinal then $\mathcal{M}$ is *κ-saturated* if for every subset X of the universe of $\mathcal{M}$ of cardinality $< \kappa$ every consistent type over X is realized in $\mathcal{M}$. We say that $\mathcal{M}$ is *saturated* if it is κ-saturated where κ is the density character of $\mathcal{M}$.

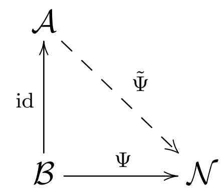
Thus countably saturated is the same as $\aleph_1$-saturated, where $\aleph_1$ is the least uncountable cardinal. The following is a version of a classical theorem of Keisler for the logic of metric structures.

Proposition 4.11. *If $\mathcal{M}_i$, for $i \in \mathbb{N}$, are models of the same language and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ then the ultraproduct $\prod_{\mathcal{U}} \mathcal{M}_i$ is countably saturated. If $\mathcal{M}$ is separable then the relative commutant of $\mathcal{M}$ in $\mathcal{M}^{\mathcal{U}}$ is countably saturated.*

Proof. A straightforward diagonalization argument, cf. the proof of Proposition 4.8. □

The following lemma is a key tool.

Lemma 4.12. *Assume $\mathcal{N}$ is a countably saturated $\mathcal{L}$-structure, $\mathcal{A}$ and $\mathcal{B}$ are separable $\mathcal{L}$-structures, and $\mathcal{B}$ is an elementary submodel of $\mathcal{A}$. Also assume $\Psi\colon \mathcal{B} \to \mathcal{N}$ is an elementary embedding. Then Ψ can be extended to an elementary embedding $\Phi\colon \mathcal{A} \to \mathcal{N}$.*



Proof. Enumerate a countable dense subset of A as a_n, for $n \in \mathbb{N}$, and fix a countable dense $B_0 \subseteq B$. Let t_n be the type of a_n over $B_0 \cup \{a_j : j < n\}$. If t is a type over a subset X of A then by $\Psi(t)$ we denote the type over the Ψ-image of X obtained from t by replacing each $a \in A$ by $\Psi(a)$. By countable saturation realize $\Psi(t_0)$ in $\mathcal{N}$ and denote the realization by $\Psi(a_0)$ in order to simplify the notation. The type $\Psi(t_1)$ is realized in $\mathcal{N}$ by an element that we denote by $\Psi(a_1)$. Continuing in this manner, we find elements $\Psi(a_n)$ in $\mathcal{N}$, for $n \in \mathbb{N}$. Since the sequence a_n, for $n \in \mathbb{N}$, is dense in A, by elementarity the map $a_n \mapsto \Psi(a_n)$ can be extended to an elementary embedding $\Phi\colon \mathcal{A} \to \mathcal{N}$ as required. □

Note that the analogue of Lemma 4.12 holds when, instead of assuming $\mathcal{A}$ to be separable, $\mathcal{N}$ is assumed to be κ-saturated for some cardinal κ greater than the density character of $\mathcal{A}$. Using a transfinite extension of Cantor's back-and-forth method, Proposition 4.7 and this analogue of Lemma 4.12 one proves the following.

Proposition 4.13. *Assume $\mathcal{L}$ is a separable language. If $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent saturated models of $\mathcal{L}$ that have the same uncountable density character then they are isomorphic.* □

For simplicity, in the following discussion we refer to tracial von Neumann algebras (C*-algebras, unitary groups of a tracial von Neumann algebra or a C*-algebra, respectively) as 'algebras.'

Corollary 4.14. *Assume the Continuum Hypothesis. If M is an algebra of density character $\leq \mathfrak{c}$ then all of its ultrapowers associated with nonprincipal ultrafilters are isomorphic. If M is separable, then all of its relative commutants in ultrapowers associated with nonprincipal ultrafilters are isomorphic.*

Proof. The Continuum Hypothesis implies that such ultrapowers are saturated and by Proposition 4.3, Proposition 4.11 and Proposition 4.13 they are all isomorphic. If M is separable, then the isomorphism between the ultrapowers can be chosen to send the diagonal copy of M in one ultrapower to the diagonal copy of M in the other ultrapower and therefore the relative commutants are isomorphic. □

It should be noted that, even in the case when the Continuum Hypothesis fails, countable saturation and a transfinite back-and-forth construction together show that ultrapowers of a fixed algebra are very similar to each other.

Corollary 4.15. *Assume M is a separable algebra and $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters on $\mathbb{N}$. Then for all separable algebras N we have the following:*

(1) *N is a subalgebra of $M^{\mathcal{U}}$ if and only if it is a subalgebra of $M^{\mathcal{V}}$;*

(2) *N is a subalgebra of $M' \cap M^{\mathcal{U}}$ if and only if N is a subalgebra of $M' \cap M^{\mathcal{V}}$.*

Proof. These classes of algebras are axiomatizable, so instead of algebras we can work with the associated models. Supposing that $\mathcal{N} \subset \mathcal{M}^{\mathcal{U}}$, apply the downward Löwenheim–Skolem theorem (Proposition 4.7) to find an elementary submodel $\mathcal{P}$ of $\mathcal{M}^{\mathcal{U}}$ whose universe contains N and the diagonal copy of M. Now consider the elementary inclusion $\mathcal{M} \subseteq \mathcal{P}$, and use Lemma 4.12 to extend the map which identifies $\mathcal{M}$ with the diagonal subalgebra of $\mathcal{M}^{\mathcal{V}}$, the latter being countably saturated by Proposition 4.11. This extension carries P onto a subalgebra of $M^{\mathcal{V}}$ and restricts to an isomorphism from N onto its image. In case M and N commute, their images in $M^{\mathcal{V}}$ do too. □

We also record a refining of the fact that the relative commutants of a separable algebra are isomorphic assuming CH.

Corollary 4.16. *Assume M, $\mathcal{V}$ and $\mathcal{U}$ are as in Corollary 4.15. Then the relative commutants $M' \cap M^{\mathcal{U}}$ and $M' \cap M^{\mathcal{V}}$ are elementarily equivalent.*

Proof. By countable saturation of ultrapowers, a type p over the copy of M inside $M^{\mathcal{U}}$ is realized if and only the same type over the copy of M inside $M^{\mathcal{V}}$ is realized. By considering only types p that extend the relative commutant type the conclusion follows. □

The conclusion of Corollary 4.16 fails when M is the C*-algebra $\mathcal{B}(\ell^2)$. By [11] CH implies $\mathcal{B}(\ell^2)' \cap \mathcal{B}(\ell^2)^{\mathcal{U}}$ is trivial for one $\mathcal{U}$ and infinite-dimensional for another. This implies that the assumption of separability is necessary in Corollary 4.16.

5. Stability, the order property, and nonisomorphic ultrapowers

This section defines the two main model theoretic notions of the paper: stability and the order property. We show that each is equivalent to the negation of the other (Theorem 5.5), and that the order property is equivalent to the existence of nonisomorphic ultrapowers when the continuum hypothesis fails (Theorem 5.6). While the analogue of the former fact is well-known in the discrete case, we could not find a reference to the analogue of the latter fact in the discrete case. We have already seen that when the continuum hypothesis holds all ultrapowers are isomorphic (Corollary 4.14). (Of course we are talking about separable structures with ultrapowers based on free ultrafilters of $\mathbb{N}$.)

5.1. Stability.

Definition 5.1. We say a theory T is **λ-stable** if for any model M of T of density character λ, the space of complete types $S(M)$ has density character λ in the metric topology on $S(M)$. We say T is **stable** if it is stable for some λ and T is **unstable** if it is not stable.

For a theory T in a separable language one can show that T is stable if and only if it is $\mathfrak{c}$-stable (see the proof of Theorem 5.5).

Our use of the term "stable" in this paper agrees with model theoretic terminology in both continuous and discrete logic. Motivated by model theory, in 1981 Krivine and Maurey defined a related notion of stability for Banach spaces that is now more familiar to many analysts ([18]). It is characterized by the requirement

$$(*) \qquad \lim_{i\to\mathcal{U}} \lim_{j\to\mathcal{V}} \|x_i + y_j\| = \lim_{j\to\mathcal{V}} \lim_{i\to\mathcal{U}} \|x_i + y_j\|$$

for any uniformly bounded sequences $\{x_i\}$ and $\{y_j\}$, and any free ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$. One can show ([16]) that a Banach space satisfies (*) if and only if no quantifier-free formula has the order property in that structure(cf. §5.2), so model theoretic stability of the theory of a Banach space X implies stability of Xin the sense of Krivine-Maurey.

We proved in [10] that all infinite-dimensional C*-algebras are unstable. The same cannot be said for infinite-dimensional Banach algebras: take a stable Banach space and put the zero product on it. However a stable Banach space can become unstable when it is turned into a Banach algebra. We exhibit this behavior in Proposition 6.2 below.

5.2. **The order property.**

Definition 5.2. We say that a continuous theory T **has the order property** if

- there is a formula $\psi(\bar{x}, \bar{y})$ with the lengths of $\bar{x}$ and $\bar{y}$ the same, and a sequence of domains $\bar{D}$ consistent with the sorts of $\bar{x}$ and $\bar{y}$, and
- a model M of T and $\langle \bar{a}_i : i \in \mathbb{N}\rangle \subseteq \bar{D}(M)$

such that

$$\psi(a_i, a_j) = 0 \text{ if } i < j \text{ and } \psi(a_i, a_j) = 1 \text{ if } i \geq j.$$

Note that these evaluations are taking place in M. Also note that by the uniform continuity of ψ, there is some $\varepsilon > 0$ such that $d(\bar{a}_i, \bar{a}_j) \geq \varepsilon$ for every $i \neq j$ where the metric here is interpreted as the supremum of the coordinatewise metrics.

Proposition 5.3. Th(A) *has the order property if and only if there is* ψ *and* $\bar{D}$ *such that for all* n *and* $\delta > 0$*, there are* $a_1, \dots, a_n \in \bar{D}(A)$ *such that*

$$\psi(a_i, a_j) \leq \delta \textit{ if } i < j \textit{ and } \psi(a_i, a_j) \geq 1 - \delta \textit{ if } i \geq j.$$

Proof. Compactness. □

Definition 5.4. Suppose that M is a metric structure and $p(\bar{x}) \in S^{\bar{D}}(M)$ is a type. We say that p is **finitely determined** if for every formula $\varphi(\bar{x}, \bar{y})$, choice of domains $\bar{D}'$ consistent with the variables $\bar{y}$, and $m \in \mathbb{N}$, there is $k \in \mathbb{N}$ and a finite set $B \subseteq \bar{D}(M)$ such that for every $\bar{c}_1, \bar{c}_2 \in \bar{D}'(M)$ (see Notation 4.9)

$$\sup_{\bar{b} \in B} |\varphi(\bar{b}, \bar{c}_1) - \varphi(\bar{b}, \bar{c}_2)| \leq \frac{1}{k} \quad \Rightarrow \quad |\varphi(p, \bar{c}_1) - \varphi(p, \bar{c}_2)| \leq \frac{1}{m}.$$

Theorem 5.5. *The following are equivalent for a continuous theory* T*:*

(1) T *is stable.*
(2) T *does not have the order property.*
(3) *If* M *is a model of* T *and* $p \in S(M)$ *then* p *is finitely determined.*

Proof. (1) implies (2) is standard: suppose that T has the order property via a formula θ and choose any cardinal λ. Fix $\mu \leq \lambda$ least such that $2^\mu > \lambda$ (note that $2^{<\mu} \leq \lambda$). By compactness, using the order property, we can find $\langle \bar{a}_i : i \in 2^{<\mu}\rangle$ such that $\theta(\bar{a}_i, \bar{a}_j) = 0$ if $i < j$ in the standard lexicographic order and 1 otherwise. Clearly, $\chi(S(A)) > \chi(A)$ where $A = \{\bar{a}_i : i \in 2^{<\mu}\}$ so T is not λ-stable for any λ.

To see that (3) implies (1), fix a model M of T with density character λ where $\lambda^{\chi(T)} = \lambda$. By assumption, every type over M is finitely determined and so there are at most $\lambda^{\chi(T)} = \lambda$ many types over M and so T is λ-stable.

Finally, to show that (2) implies (3), suppose that there is a type over a model of T which is not finitely determined. Fix $p(\bar{x}) \in S^{\bar{D}}(M)$, $\varphi(\bar{x}, \bar{y})$, domains $\bar{D}'$ consistent with the variables $\bar{y}$ and $m \in \mathbb{N}$ so that for all k and finite $B \subseteq \bar{D}(M)$, there are $n_1, n_2 \in \bar{D}'(M)$ such that

$$\max_{b \in B} |\varphi(b, n_1) - \varphi(b, n_2)| \leq 1/k$$

but

$$|\varphi(p, n_1) - \varphi(p, n_2)| > 1/m.$$

We now use this p to construct an ordered sequence. Define sequences a_j, b_j, c_j and sets B_j as follows: $B_0 = \emptyset$. If we have defined B_j, choose $b_j, c_j \subseteq \bar{D}'(M)$ such that $\max_{b \in B_j} |\varphi(b, b_j) - \varphi(b, c_j)| \leq 1/2m$ but $|\varphi(p, b_j) - \varphi(p, c_j)| > 1/m$.

Now choose $a_j \in \bar{D}(M)$ so that a_j realizes $\varphi(\bar{x}, b_i) = \varphi(p, b_i)$ and $\varphi(\bar{x}, c_i) = \varphi(p, c_i)$ for all $i \leq j$. Let $B_{j+1} = B_j \cup \{a_j, b_j, c_j\}$. It follows that if $i \geq j$ then $|\varphi(a_i, b_j) - \varphi(a_i, c_j)| > 1/m$. If $i < j$ then $|\varphi(a_i, b_j) - \varphi(a_i, c_j)| \leq 1/2m$ since $a_i \in B_j$. Consider the formula

$$\theta(x_1, y_1, z_1, x_2, y_2, z_2) := |\varphi(x_1, y_2) - \varphi(x_1, z_2)|.$$

Then θ orders $\langle a_i, b_i, c_i : i \in \mathbb{N} \rangle$. □

Theorem 5.6. *Suppose that A is a separable metric structure in a separable language.*

(1) *If the theory of A is stable then for any two non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, $A^{\mathcal{U}} \cong A^{\mathcal{V}}$.*
(2) *If the theory of A is unstable then the following are equivalent:*
  (a) *A has fewer than $2^{2^{\aleph_0}}$ nonisomorphic ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$.*
  (b) *for any two non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, $A^{\mathcal{U}} \cong A^{\mathcal{V}}$;*
  (c) *the Continuum Hypothesis holds.*

It is worth mentioning that Theorem 5.6 is true in the first order context, as can be seen by considering a model of a first-order theory as a metric model with respect to the discrete metric. Although this is undoubtedly known to many, we were unable to find a direct reference. The proof of (1) will use tools from stability theory, and the reader may want to refer to [3] for background.

Proof. (1) Assume that the theory of A is stable. We will show that $A^{\mathcal{U}}$ is $\mathfrak{c}$-saturated and so it will follow that $A^{\mathcal{U}} \cong A^{\mathcal{V}}$ no matter what the size of the continuum is (see Proposition 4.13).

So suppose that $B \subseteq A^{\mathcal{U}}$, $|B| < \mathfrak{c}$, and q is a type over B. We may assume that B is an elementary submodel and that q is nonprincipal and complete. As the theory of A is stable, choose a countable elementary submodel $B_0 \subseteq B$ so that q does not fork over B_0. We shall show that in $A^{\mathcal{U}}$ one can always find a Morley sequence in $q|_{B_0}$ of size $\mathfrak{c}$.

Towards this end, fix a countable Morley sequence I in the type of $q|B_0$ and let $\bar{q} = tp(I/B_0)$, a type in the variables x_n for $n \in \mathbb{N}$. Since B_0 is countable and the language is separable, there are countably many formulas $\psi_n(x_1, \ldots, x_n, b_n)$ over B_0 such that $\psi_n(x_1, \ldots, x_n, b_n) = 0 \in \bar{q}$ and $\{\psi_n(x_1, \ldots, x_n, b_n) = 0 : n \in \mathbb{N}\}$ axiomatizes $\bar{q}$.

Now let $D_i = \{n \geq i : \inf_x \psi_i(x, b_i(n)) < 1/i\}$ For a fixed n, consider $\{i : n \in D_i\}$. This set has a maximum element; call it i_n. Now fix $a^n_1, \ldots, a^n_{i_n} \in A$ such that $\psi_{i_n}(a^n_1, \ldots, a^n_{i_n}, b_{i_n}) < 1/i_n$. Now consider the set J of all $g : \mathbb{N} \to A$ such that $g(n) \in \{a^n_1, \ldots, a^n_{i_n}\}$ for all n. Any $g \in J$ will satisfy $q|B_0$ in $A^{\mathcal{U}}$ since every element of I realized that type. If $g_0, \ldots, g_k$

are in J and distinct modulo U then they are independent over B_0 since I was a Morley sequence. To finish then, we need to see that there are $\mathfrak{c}$-many distinct g's modulo U. This follows from the fact that the i_n's are not bounded. To see this, for a fixed m, let $X = \{n \geq m : \inf_x \psi_m(x, b(n)) \leq 1/m\}$. Pick any $n \in X$. We have that $n \in D_m$ so $i_n \geq m$ and we conclude that the i_n's are not bounded.

Since $|B| < \mathfrak{c}$, there is a J_0 of cardinality less than $\mathfrak{c}$ such that B is independent from J over J_0. Choosing any $a \in J \setminus J_0$ and using symmetry of non-forking remembering that J is a Morley sequence over B_0, it follows that a is independent from B over B_0. Since B_0 is a model, $q|_{B_0}$ has a unique non-forking extension to B and it follows that a realizes q. This finishes the proof of (1).

(2) If the Continuum Hypothesis holds then $A^{\mathcal{U}}$ is always saturated and so for any two ultrafilters $\mathcal{U}, \mathcal{V}$, $A^{\mathcal{U}} \cong A^{\mathcal{V}}$ even if we fix the embedded copy of A (Corollary 4.14).

The implication (a) implies (c) follows from [12, Theorem 3] and of course (b) implies (a). However, (b) implies (c) also can be proved by a minor modification of proof from [10] (which is in turn a modification of a proof from [8]), so we assume that the reader has a copy of the former handy and we sketch the differences. Assume the theory of A is unstable. Then by Theorem 5.5 it has the order property. The formula ψ witnessing the order property satisfies [10, Properties 2.1] by Corollary 4.4. Therefore the analogues of [10, Lemma 2.4, Lemma 2.5 and Proposition 2.6] can be proved by quoting their proofs verbatim. Hence if $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ then $\kappa(\mathcal{U}) = \lambda$ (defined in [10, paragraph before Lemma 2.5]) if and only if there is a $(\aleph_0, \lambda)$-ψ-gap in $A^{\mathcal{U}}$.

By [8, Theorem 2.2], if CH fails then there are ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $\kappa(\mathcal{U}) \neq \kappa(\mathcal{V})$ and this concludes the proof. □

6. Concluding Remarks

In this final section we include two examples promised earlier and state three rather different problems.

6.1. **A non-axiomatizable category of C*-algebras.** Recall that UHF, or uniformly hyperfinite, algebras are C*-algebras that are C*-tensor products of (finite-dimensional) matrix algebras. They form a subcategory of C*-algebras and the morphisms between them are *-homomorphisms.

Proposition 6.1. *The category of UHF algebras is not axiomatizable.*

Proof. By Proposition 4.1 it will suffice to show that this category is not closed under taking (C*-algebraic) ultraproducts. We do this by repeating an argument from Ge-Hadwin ([13, Corollary 5.5]) exploiting the fact that UHF algebras have unique traces that are automatically faithful.

Let A be the CAR algebra $\bigotimes_{n\in\mathbb{N}} M_2(\mathbb{C})$ with trace tr, let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and let $\{p_n\} \subset A$ be projections with $tr(p_n) = 2^{-n}$. The sequence (p_n) represents a nonzero projection in $A^{\mathcal{U}}$, but $tr^{\mathcal{U}}((p_n)) = 0$. Thus $tr^{\mathcal{U}}$ is a non-faithful tracial state, so that $A^{\mathcal{U}}$ is not UHF. □

The same argument shows that simple C*-algebras are not axiomatizable.

Since every UHF algebra has a unique trace one could also consider tracial ultraproducts, instead of norm-ultraproducts, of UHF algebras. However, such an ultraproduct is always a

II_1 factor ([15, Theorem 4.1]) and therefore not a UHF algebra (because projections in UHF algebras have rational traces).

6.2. **An unstable Banach algebra whose underlying Banach space is stable.** The L^p Banach spaces ($1 \le p < \infty$) are known to be stable ([1, Section 17]), and they become stable Banach algebras when endowed with the zero product. Actually ℓ^p ($1 \le p < \infty$) with pointwise multiplication is also stable; this can be shown by methods similar to [10, Lemma 4.5 and Proposition 4.6]. We now prove that the usual convolution product turns $\ell^1(\mathbb{Z})$ into an unstable Banach algebra, as was mentioned in §5.1.

Proposition 6.2. *The Banach algebra $\ell^1 = \ell^1(\mathbb{Z}, +)$ (with convolution product) is unstable.*

Proof. It suffices to show the order property for ℓ^1. This means we must give a formula $\varphi(x, y)$ of two variables (or n-tuples) on ℓ^1, a bounded sequence $\{x_i\} \subset \ell^1$, and $\varepsilon > 0$ such that $\varphi(x_i, x_j) \le \varepsilon$ when $i \le j$ and $\varphi(x_i, x_j) \ge 2\varepsilon$ when $i > j$.

Let $\{f_n\}_{n\in\mathbb{Z}}$ denote the standard basis for ℓ^1, so that multiplication is governed by the rule $f_m f_n = f_{m+n}$. Also let $\ell^1 \ni x \mapsto \hat{x} \in C(\mathbb{T})$ be the Gel'fand transform on ℓ^1, so that $\hat{f}_n$ is the function $[e^{it} \mapsto e^{int}]$. The Gel'fand transform is always a contractive homomorphism; on ℓ^1 it is injective but not isometric.

We take $\varphi(x, y) = \inf_{\|z\|\le 1} \|xz - y\|$, $x_i = (\frac{f_1+f_{-1}}{2})^{2^i}$, and $\varepsilon = \frac{1}{8}$. Note that all the x_i are unit vectors, being convolution powers of a probability measure on $\mathbb{Z}$, and $\hat{x}_i = [e^{it} \mapsto (\cos t)^{2^i}] \in C(\mathbb{T})$.

For $i \le j$, we have $\varphi(x_i, x_j) = 0$ by taking $z = (\frac{f_1+f_{-1}}{2})^{2^j-2^i}$.

For $i > j$, let $t_0 \in (0, 2\pi)$ be such that $(\cos t_0)^{2^j} = \frac{1}{2}$. For any $z \in (\ell^1)_{\le 1}$,

$$\|x_i z - x_j\|_{\ell^1} \ge \|\hat{x}_i \hat{z} - \hat{x}_j\|_{C(\mathbb{T})} \ge \left| \left(\frac{1}{2}\right)^{2^{i-j}} \hat{z}(e^{it_0}) - \frac{1}{2} \right| \ge \frac{1}{4},$$

where the middle inequality is justified by evaluation at t_0. We conclude that $\varphi(x_i, x_j) \ge \frac{1}{4}$ as desired. □

Question 6.3. *Is there a nice characterization of stability for Banach algebras?*

It is obviously necessary that the underlying Banach space be stable. The proof of Proposition 6.2 works for any abelian Banach algebra that contains a unit vector g such that the range of $\hat{g}$ contains nonunit scalars with modulus arbitrarily close to 1. (Above, g was $\frac{f_1+f_{-1}}{2}$.) So for instance the convolution algebra $L^1(\mathbb{R}, +)$ is covered, and in fact *any* probability density will do for g in this case.

6.3. **K-theory reversing automorphisms of the Calkin algebra.** A well-known problem of Brown–Douglas–Fillmore ([5, 1.6(ii)]) asks whether there is an automorphism of the Calkin algebra that sends the image of the unilateral shift to its adjoint. The main result of [9] implies that if ZFC is consistent then there is a model of ZFC in which there is no such automorphism. A deep metamathematical result of Woodin, known as the Σ^2_1-*absoluteness theorem*, essentially (but not literally) implies that the Brown–Douglas–Fillmore question has a positive answer if and only if the Continuum Hypothesis implies a positive answer (see [25]). The type referred to in the following question is the type over the empty set in the sense of §4.3.

Question 6.4. *Do the image of the unilateral shift in the Calkin algebra and its adjoint have the same type in the Calkin algebra?*

A negative answer to Question 6.4 would imply a negative answer to the Brown–Douglas–Fillmore problem. A positive answer to the former would reduce the latter to a problem about model-theoretic properties of the Calkin algebra. The Calkin algebra is not countably saturated (first author, unpublished), but it may be countably homogeneous. For example, its poset of projections is countably saturated ([14], see also [20]). Countable homogeneity of the Calkin algebra would, in the presence of the Continuum Hypothesis, imply that Question 6.4 has a positive answer if and only if the Brown–Douglas–Fillmore question has a positive answer. It is not difficult to prove that the (model-theoretic) types of operators in the Calkin algebra, and therefore the answer to Question 6.4, are absolute between transitive models of ZFC.

6.4. **Matrix algebras.** We end with discussion of finite-dimensional matrix algebras and a result that partially complements [10, Proposition 3.3], where it was proved that if the Continuum Hypothesis fails then the matrix algebras $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, have nonisomorphic tracial ultraproducts.

Proposition 6.5. *Every increasing sequence $n(i)$, for $i \in \mathbb{N}$, of natural numbers has a further subsequence $m(i)$, for $i \in \mathbb{N}$ such that if the Continuum Hypothesis holds then all tracial ultraproducts of $M_{m(i)}(\mathbb{C})$, for $i \in \mathbb{N}$, are isomorphic.*

Proof. The set of all $\mathcal{L}$-sentences (see §2) is separable. Let $\mathbf{T}_n = \mathrm{Th}(M_n(\mathbb{C}))$, the map associating the value $\psi^{M_n(\mathbb{C})}$ of a sentence ψ in $M_n(\mathbb{C})$ to ψ. Since the set of sentences is separable we can pick a sequence $m(i)$ so that the theories $\mathbf{T}_{m(i)}$ converge pointwise to some theory $\mathbf{T}_\infty$. Let $\mathcal{U}$ be a nonprincipal ultrafilter such that $\{m(i) : i \in \mathbb{N}\} \in \mathcal{U}$. By Łos's theorem, Proposition 4.3, $\mathrm{Th}(\prod_{\mathcal{U}} M_n(\mathbb{C})) = \mathbf{T}_\infty$. By Proposition 4.11 and the Continuum Hypothesis all such ultrapowers are saturated and therefore Proposition 4.13 implies all such ultrapowers are isomorphic. □

Question 6.6. *Let ψ be an $\mathcal{L}$-sentence. Does $\lim_{n\to\infty} \psi^{M_n(\mathbb{C})}$ exist?*

A positive answer is equivalent to the assertion that all ultraproducts $\prod_{\mathcal{U}} M_n(\mathbb{C})$ are elementarily equivalent, and therefore isomorphic if the Continuum Hypothesis is assumed (see Proposition 4.11 and Proposition 4.13).

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