Minimum-Effort Coordination Games: Stochastic Potential and Logit Equilibrium*

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ABSTRACT

This paper revisits the minimum-effort coordination game with a continuum of Pareto-ranked Nash equilibria. Noise is introduced via a logit probabilistic choice function. The resulting logit equilibrium distribution of decisions is unique and maximizes a stochastic potential function. In the limit as the noise vanishes, the distribution converges to an outcome that is analogous to the risk-dominant outcome for 2×2 games. In accordance with experimental evidence, logit equilibrium efforts decrease with increases in effort costs and the number of players, even though these parameters do not affect the Nash equilibria.

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I. Introduction

There is a widespread interest in coordination games with multiple Pareto-ranked equilibria, since these games have equilibria that are bad for all concerned. The coordination game is a particularly important paradigm for those macroeconomists who believe that an economy may become mired in a low-output equilibrium (e.g., Bryant, 1983, Cooper and John, 1988, and Romer, 1996, section 6.14). Coordination problems can be solved by markets in some contexts, but market signals are not always available. For example, if a high output requires high work efforts by all members of a production team, it may be optimal for an individual to shirk when others are expected to do the same. In the minimum-effort coordination game, which results from perfect complementarity of players' effort levels, *any* common effort constitutes a Nash equilibrium. Without further refinement, the Nash equilibrium concept provides little predictive power. Moreover, the set of equilibria is unaffected by changes in the number of participants or the cost of effort, whereas intuition suggests that efforts should be lower when effort is more costly, or when there are more players (Camerer, 1997). The dilemma for an individual is that better outcomes require higher effort but entail more risk. Uncertainty about others' actions is a central element of such situations.

Motivated by the observation that human decisions exhibit some randomness, we introduce some noise in the decision-making process, in a manner that generalizes the notion of a Nash equilibrium. Our analysis is an application of the approach developed by Rosenthal (1989) and McKelvey and Palfrey (1995). We extend their analysis to a game with a continuum of actions and use the logit probabilistic choice framework to determine a "logit equilibrium," which determines a *unique* probability distribution of decisions in a coordination game that has a continuum of pure-strategy Nash equilibria. We then analyze the comparative static properties of the logit equilibrium for the minimum-effort game and compare these theoretical properties with experimental data.[1]

Van Huyck, Battalio, and Beil (1990) have conducted laboratory experiments with a minimum-effort structure, with seven effort levels and seven corresponding Pareto-ranked Nash equilibria in pure strategies (regardless of the number of players). The intuition that coordination is more difficult with more players is apparent in the data: behavior in the final periods typically approaches the "worst" Nash outcome with a large number of players, whereas the "best" equilibrium has more drawing power with two players. An extreme reduction in the cost of effort (to zero) results in a preponderance of high-effort decisions. Goeree and Holt (1998) also report results for a minimum-effort coordination game experiment, but with a continuum of decisions and non-extreme parameter choices. Effort distributions tend to stabilize after several periods of random matching, and there is a sharp inverse relationship between effort costs and average effort levels.

The most salient features of these experimental results cannot be explained by a Nash analysis, since the set of Nash equilibria is unaffected by changes in the effort cost or the number of players. This invariance is caused by the fact that best responses used to construct a Nash equilibrium depend on the signs, not magnitudes, of payoff differences. In particular, best responses in a minimum-effort game do not depend on noncritical changes in the effort cost or the number of players, but magnitudes of payoff differences do. When effort costs are low and others' behavior is noisy, exerting a lot of effort yields high payoffs when others do so too, and exerting a lot of effort is not too costly when others shirk. The high expected payoff that results from high efforts is reflected in the logit equilibrium density which puts more probability mass at high efforts, which in turn reinforces the payoff from exerting a lot of effort. Likewise, with a large number of players any noise in the decisions tends to result in low minimum efforts, which raises the risk of exerting a high effort. The logit equilibrium formalizes the notion that asymmetric risks can have large effects on behavior when there is some noise in the system.

There has, of course, been considerable theoretical work on equilibrium selection in coordination games, although most of this work concerns 2×2 games. Most prominent here is the Harsanyi and Selten (1988) notion of "risk dominance," which captures the tradeoff between high payoffs and high risk. The risk-dominant Nash equilibrium for a 2×2 game is the one that minimizes the product of the players' losses associated with unilateral deviations. Game theorists have interpreted risk dominance as an appealing selection criterion in need of a sound theoretical underpinning. For instance, Carlsson and van Damme (1993) assume that players make noisy observations of the true payoffs in a 2×2 game. They show that in the limit as this "measurement error" disappears, iterated elimination of dominated strategies requires players to make decisions that conform to the risk-dominant equilibrium. Alternatively, Kandori, Mailath, and Rob (1993) and Young (1993) specify noisy models of evolution, and show that behavior converges to the risk-dominant equilibrium in the limit as the noise vanishes.

These justifications of risk dominance are limited to simple 2×2 games, and there is no general agreement on how to generalize risk dominance to broader classes of games. However, it is well known that the risk-dominant outcome in a 2×2 coordination game coincides with the one that maximizes the "potential" of the game (e.g. Young, 1993). Loosely speaking, the potential of a game is a function of all players' decisions which increases with unilateral changes that increase a player's payoffs. Thus any Nash equilibrium is a stationary point of the potential function (Rosenthal, 1973; Monderer and Shapley, 1996). The intuition behind potential is that if each player is moving in the direction of higher payoffs, each of the individual movements will raise the value of the potential, which ends up being maximized in equilibrium. This notion of a potential function does generalize to a broader class of games, including the continuous coordination game considered in this paper. Monderer and Shapley (1996) have already proposed using the potential function as a refinement device for the coordination game to explain the experimental results of Van Huyck, Battalio, and Beil (1990). However, they "do not attempt to provide any explanation to this prediction power obtained (perhaps as a coincidence) in this case" (Monderer and Shapley, 1996, p.126-127). Our results indicate why this refinement might work reasonably well. Specifically, we prove that the logit equilibrium selects the distribution that is the maximum of a *stochastic potential*, which is obtained by adding a measure of dispersion (entropy) to the expected value of the standard potential. Thus the logit equilibrium, which maximizes stochastic potential, will also tend to maximize ordinary potential in low-noise environments.[2] An econometric analysis of laboratory data, however, indicates that the best fits are obtained with noise parameters that are significantly different from zero, even in the final periods of coordination game experiments (Goeree and Holt, 1998).

The next section specifies the minimum effort game structure and the equilibrium concept. Symmetry and uniqueness properties are proved in section III. The fourth section derives the effects of changes in the effort cost and the number of players and derives the limit equilibrium as the noise vanishes. Section V contains a discussion of potential, stochastic potential, and risk-dominance for the minimum-effort game, and shows that the logit equilibrium is a stationary point of the stochastic potential. The final section summarizes.

II. The Minimum Effort-Coordination Game

Consider an *n*-person coordination game in which each player *i* chooses an effort level, x_i, $i = 1,..., n$. Production has a "team" structure when each player's effort increases the marginal products of one or more of the others' effort inputs. Here, we consider the extreme case in which efforts are perfect complements: the common part of the payoff is determined by the *minimum* of the *n* effort levels.[3] Each player's payoff equals the difference between the common payoff and the (linear) cost of that player's own effort, so:

$$\pi_i(x_1, ..., x_n) = \min_{j=1,...,n} \{x_j\} - c\, x_i, \qquad i = 1, ..., n, \tag{1}$$

and each player chooses an effort from the interval $[0, \bar{x}]$. The problem is interesting when the marginal per capita benefit from a coordinated effort increase, 1, is greater than the marginal cost, and therefore, we assume $0 < c < 1$. The important feature of this game is that *any common effort level is a Nash equilibrium,* since a costly unilateral increase in effort will not affect the minimum effort, while a unilateral decrease reduces the minimum by more than the cost saving. Therefore, the payoff structure in (1) produces a continuum of pure-strategy Nash equilibria. These equilibria are Pareto-ranked because all individuals prefer an equilibrium with higher effort levels for all. As shown in the Appendix, there is also a continuum of (Pareto-ranked) symmetric mixed-strategy Nash equilibria. These equilibria have unintuitive comparative static properties in the sense that increases in the effort cost or in the number of players *increase* the expected effort.

In practice, the environments in which individuals interact are rarely so clearly defined as in (1). Even in experimental set-ups, in which money payoffs can be precisely stated, there is still some residual haziness in the players' actual objectives, in their perceptions of the payoffs, and in their reasoning. These considerations motivate us to model the decision process as inherently noisy from the perspective of an outside observer. We use a continuous analogue of the standard (logit) probabilistic choice framework, in which the probability of choosing a decision is proportional to an exponential function of the observed payoff for that decision. The standard derivation of the logit model is based on the assumption that payoffs are subject to unobserved preference shocks from a double-exponential distribution, e.g. Anderson, de Palma, and Thisse (1992).[4] When the set of feasible choices is an interval on the real line, player *i*'s probability density is an exponential function of the expected payoff, $\pi_i^e(x)$:

$$f_i(x) = \frac{\exp(\pi_i^e(x)/\mu)}{\int_0^{\bar{x}} \exp(\pi_i^e(s)/\mu)\, ds}, \qquad i = 1, \ldots, n, \tag{2}$$

where $\mu > 0$ is the noise parameter. The denominator on the right hand side of (2) is a constant, independent of *x*, and ensures that the density integrates to 1: since $\pi_i^e(0) = 0$ for the minimum effort game, the denominator is $1/f_i(0)$, and (2) can be written as $f_i(x) = f_i(0)\exp(\pi_i^e(x)/\mu)$. The sensitivity of the density to payoffs is determined by the noise parameter. As $\mu \to 0$, the probability of choosing an action with the highest expected payoff goes to one. Higher values of μ correspond to more noise: if μ tends to infinity, the density function in (2) becomes flat over its whole support and behavior becomes random.

Equation (2) has to be interpreted carefully because the choice density that appears on the left is also used to determine the expected payoffs on the right. The *logit equilibrium* is a vector of densities that is a fixed point of (2) (McKelvey and Palfrey, 1995).[5] The next step is to apply the probabilistic choice rule (2) to the payoff structure in (1).

III. Equilibrium Effort Distributions

The equilibrium to be determined is a probability density over effort levels. We first derive the integral/differential equations that the equilibrium densities, $f_i(x)$, must satisfy. These equations are used to prove that the equilibrium distribution is the same for all players and is unique. Although we can find explicit solutions for the equilibrium density for some special cases, the general symmetry and uniqueness propositions are proved by contradiction, a method that is quite useful in applications of the logit model. The proofs can be skipped on a first reading. The uniqueness of the equilibrium is a striking result given the continuum of Nash equilibria for the payoff structure in (1).

For an individual player, the relevant statistic regarding others' decisions is summarized by the distribution of the minimum of the n-1 other effort levels. For individual i, this distribution is represented by $G_i(x)$, with density $g_i(x)$. The probability that the minimum of others' efforts is below x is just one minus the probability that all other efforts are above x, so $G_i(x) = 1 - \prod_{k \neq i}(1 - F_k(x))$, where $F_k(x)$ is the effort distribution of player k. Each player's payoff is the minimum effort, minus the cost of the player's own effort (see (1)). Thus player i's expected payoff from choosing effort level, x, is:

$$\pi_i^e(x) \;=\; \int_0^x y\, g_i(y)\, dy \;+\; x\,(1 - G_i(x)) \;-\; c\,x, \qquad i = 1,\ldots,n, \tag{3}$$

where the first term on the right side is the benefit when some other player's effort is below the player's own effort, x, and the second term is the benefit when player i determines the minimum effort. The right side of (3) can be integrated by parts to obtain:

$$\pi_i^e(x) \;=\; \int_0^x (1 - G_i(y))\, dy \;-\; c\,x \;=\; \int_0^x \prod_{k \neq i} (1 - F_i(y))\, dy - c\,x. \tag{4}$$

where the second equality follows from the definition of $G_i(\cdot)$. The expected payoff function in (4) determines the optimal decision as well as the cost of deviating from the optimum. Such deviations can result from unobserved preference shocks. The logit probabilistic choice function in (2) ensures that more costly deviations are less likely.

The first issue to be considered is existence of a logit equilibrium. McKelvey and Palfrey (1995) prove existence of a (more general) quantal response equilibrium for finite normal-form games. However, their proof does not cover continuous games such as the minimum-effort coordination game considered in this paper.

Proposition 1. There exists a logit equilibrium for the minimum-effort coordination game. Furthermore, each player's effort density is differentiable at any logit equilibrium.

Proof. Monderer and Shapley (1996) show that the minimum effort game is a potential game (see also Section V). Anderson, Goeree, and Holt (1997, Proposition 3, Corollary 1) prove that a logit equilibrium exists for any continuous potential game when the strategy space is bounded. Thus an equilibrium exists for the present game. Now consider differentiability. Each player's expected payoff function in (4) is a continuous function of x for any vector of distributions of the others' efforts. A player's effort density is an exponential transformation of expected payoff, and hence each density is a continuous function of x as well. Therefore the distribution functions are continuous, and the expected payoffs in (4) are differentiable. The effort densities in (2) are exponential transformations of expected payoffs, and so these densities are also differentiable. Thus all vectors of densities get mapped into vectors of differentiable densities, and any fixed point must be a vector of differentiable density functions. Q.E.D.

Next we consider symmetry and uniqueness properties of the logit equilibrium. Differentiating both sides of (2) with respect to x shows that the slope of the density agrees in sign with the slope of the expected payoff function: $f_i'(x) = f_i(x)\ \pi_i^{e\prime}(x)/\mu$, where the primes denote derivatives with respect to x. The derivative of the expected payoff in (4) is then used to obtain:

$$f_i^{\prime}(x) \;=\; f_i(x)\left(\Pi_{k \neq i}(1 - F_k(x)) \;-\; c\right)/\mu, \qquad i = 1,\ldots,n, \tag{5}$$

which yields a vector of differential equations in the equilibrium densities. Given the symmetry of the model and the symmetric structure of the Nash equilibria, it is not surprising that the logit equilibrium is symmetric.

Proposition 2. Any logit equilibrium for the minimum-effort coordination game is symmetric across players, i.e. F_i is the same for all i.

Proof. Suppose (in contradiction) that the equilibrium densities for players i and j are different. In particular, $f_i(x) = f_j(x)$ for $x < x_a$, but without loss of generality $f_i(x) > f_j(x)$ on some interval $(x_a,\ x_b)$. (Note that x_a may be 0.) By Proposition 1, the densities are continuous and must integrate to 1, so they must be equal at some higher value, x_b, with $f_i(x)$ approaching $f_j(x)$ from above as x tends to x_b. Thus $f_i(x_b) = f_j(x_b)$, $f_i'(x_b) \leq f_j'(x_b)$, and $F_i(x_b) > F_j(x_b)$. Notice that F_j

appears in the product on the right side of (5) that determines the slope of f_i, and F_i appears in the product on the right side of (5) that determines the slope of f_j. By hypothesis, $1 - F_j(x_b) > 1 - F_i(x_b)$, and hence $\prod_{k\neq i}(1 - F_k(x_b)) > \prod_{k\neq j}(1 - F_k(x_b))$. Then (5) implies that $f_i'(x_b) > f_j'(x_b)$, which contradicts the requirement that the density for player *i* crosses the other density from above. Q.E.D.

Given symmetry, we can drop the *i* subscripts from (5) and write the common density as

$$f'(x) = (1 - F(x))^{n-1} f(x)/\mu - c\,f(x)/\mu. \tag{6}$$

The useful feature of this equation is that it can be integrated to derive a characterization of the equilibrium that has a different form from (2). Indeed, integrating both sides of (6) from 0 to *x* and using the condition that $F(0) = 0$ yields:

$$f(x) = f(0) + \frac{1}{n\mu}\left(1 - (1 - F(x))^n\right) - \frac{c}{\mu}F(x), \tag{7}$$

which is a first-order differential equation in the equilibrium distribution, and plays a key role in the analysis that follows. For some special cases we can find reduced forms for the relation in (7). These closed-form solutions can be useful in constructing likelihood functions for econometric tests of the theory, perhaps using laboratory data: they also help with the analysis of the general model by indicating the type of distribution that constitutes a logit equilibrium.

When there are only two players and the effort cost $c = 1/2$, equation (7) reduces to

$$f(x) = f(0) + \frac{1}{2\mu}F(x)(1 - F(x)),$$

which clearly yields a density that is symmetric around the median (where $F(x) = 1 - F(x)$), i.e. at $x = \bar{x}/2$. This equation is the defining characteristic of a logistic distribution.[6] The logistic form does not rely on the restriction to $c = ½$. Indeed, the equilibrium distribution for $n = 2$ is a truncated logistic:

$$F(x) = \frac{B}{1 + \exp\left(-B(x - M)/2\mu\right)} - \frac{B}{2} + 1 - c, \tag{8}$$

where B and M are determined by the boundary conditions $F(0) = 0$ and $F(\bar{x}) = 1$. It is straightforward to verify that (8) satisfies (7) for any values of B and M, which in turn are determined by the boundary conditions.[7] The parameterization in (8) is useful, since the location parameter M is the mode of the distribution, as can be shown by equating the second derivative of (8) to zero. When the noise parameter μ goes to 0, the distribution function has a "step" at $x = M$.

Even though the equilibrium is logistic for the two-player case, we can determine a closed-form solution for $n > 2$ only when there is no upper bound on effort.[8] Nonetheless, we can still characterize the properties of the logit equilibrium. First of all, the solution to equation (7) is unique, as the following proposition demonstrates.

Proposition 3. The logit equilibrium for the minimum-effort coordination game is unique.

Proof. Since any equilibrium must be symmetric, suppose that there are two symmetric equilibria. Equation (7) is a first-order differential equation, so the equilibrium densities are completely determined by (7) and their values at $x = 0$. Hence, the two symmetric equilibrium densities can only be different at some point when their values at $x = 0$ differ. Let the candidate distributions be distinguished with "I" and "II" subscripts, and suppose without loss of generality that $f_I(0)/f_{II}(0) > 1$, so $F_I(x)$ exceeds $F_{II}(x)$ for small enough $x > 0$. These distribution functions will converge eventually, since they must be equal at the upper bound of the support, if not before. Let x_c denote the lowest value of x at which they are equal, so $F_I(x_c) = F_{II}(x_c)$ and $f_I(x_c) \leq f_{II}(x_c)$. At x_c, all terms involving the distribution function on the right side of (7) are equal for the two distributions. Since $f_I(0) > f_{II}(0)$, it follows that $f_I(x_c) > f_{II}(x_c)$, which contradicts the fact that $F_I(x_c)$ must not have a higher slope than $F_{II}(x_c)$. Q.E.D.

This uniqueness result is surprising because an arbitrarily small amount of noise (μ > 0) shrinks the set of symmetric Nash equilibria from a continuum of pure-strategy equilibria to a single distribution. The continuum of Nash equilibria arises because the best-response functions overlap. The introduction of a small amount of noise perturbs the (probabilistic) best response functions thereby yielding a unique equilibrium distribution of effort decisions.

It would be somewhat misleading, however, to view this approach as providing a general equilibrium selection mechanism that always picks a unique outcome. Indeed, there are other games in which the logit equilibrium is not unique (see McKelvey and Palfrey, 1995). In the coordination game (1) the continuum of Nash equilibria is due to the linearity of the payoff structure, and it would also be possible to recover a unique Nash equilibrium by adding appropriate non-linearities.[9] We chose instead to keep the linear structure and incorporate some noise, since it is uncertainty about others' decisions that makes coordination problems interesting. As we show next, this modelling description yields richer predictions.

IV. Properties of the Equilibrium Effort Distribution

Intuitively, one would expect that higher effort costs and more players would make it more difficult to coordinate on preferred high-effort outcomes, even though the set of pure-strategy Nash outcomes is not affected by these parameters. This intuition is borne out by the next proposition.

Proposition 4. Increases in c and n result in lower equilibrium efforts (in the sense of first-degree stochastic dominance).

Proof. First consider a change in n, and let $F_1(x)$ and $F_2(x)$ denote the equilibrium distributions for n_1 and n_2, where $n_1 > n_2$. Suppose that $F_1(x) = F_2(x)$ on some interval of x values. Then the first two derivatives of these functions must be equal on the interval, which is impossible by (6). Thus the distribution functions can only be equal, or cross, at isolated points. At any crossing, $F_1(x) = F_2(x) \equiv F$. Since the effort cost is the same, it follows from (7) that the difference in slopes at the crossing is

$$f_1(x) - f_2(x) = f_1(0) - f_2(0) + \frac{1-(1-F)^{n_1}}{n_1\mu} - \frac{1-(1-F)^{n_2}}{n_2\mu}, \qquad (9)$$

where $f_i(x)$ denotes the density associated with n_i, i=1,2. Since $n_1 > n_2$, it is straightforward to show that the right side of (9) is decreasing in F (and hence it is decreasing in x). It follows that there can be at most two crossings, with the sign of the right-hand side non-negative at the first crossing and non-positive at the second. Since the distributions cross at $x = 0$ and $x = \bar{x}$, these are the only crossings. (There cannot be three crossings, with the right side of (9) positive at $x = 0$, zero at some interior x^*, and negative at $x = \bar{x}$, i.e. a tangency of the distribution functions at x^*. Such a tangency would require that $f_1'(x^*) \geq f_2'(x^*)$, which is impossible by (6).) The right side of (9) is positive at $x = 0$ or negative at $x = \bar{x}$, so $F_1(x) > F_2(x)$ for all interior x. The proof for the effect of a cost increase, $c_1 > c_2$, is analogous. The resulting distributions, again denoted by $F_1(x)$ and $F_2(x)$, cannot be equal on some interval without violating (6). With different costs and equal values of n, equation (7) yields: $f_1(x) - f_2(x) = f_1(0) - f_2(0) + (c_2 - c_1)F/\mu$, which is decreasing in F, and hence in x. It follows that these distributions can only cross twice, at the end points, with $f_1(0) > f_2(0)$ or $f_1(\bar{x}) < f_2(\bar{x})$, and therefore $F_1(x) > F_2(x)$ for all interior x. Q.E.D.

One possible treatment of interest in a laboratory experiment is to increase all payoffs to get subjects to consider their decisions more carefully. It follows from (2) that multiplying all payoffs by a factor K is equivalent to dividing μ by K. This raises the issue of what happens in the limiting case as the noise vanishes, which is of interest since it corresponds to perfect rationality. The following proposition characterizes the (unique) Nash equilibrium that is obtained as μ goes to zero in the logit equilibrium. Since there is a continuum of Nash equilibria for this game, this result shows that not all Nash equilibria can be attained as limits of a logit equilibrium.[10]

Proposition 5. As the noise parameter, μ, is reduced to zero, the equilibrium density converges to a point-mass at $\bar{x}$ if $c < 1/n$, at $\bar{x}/n$ if $c = 1/n$, and at 0 if $c > 1/n$.

Proof. First, consider the case $c < 1/n$. We have to show that $F(x) = 0$ for $x < \bar{x}$. Suppose not, and $F(x) > 0$ for $x \in (x_a, x_b)$. From equation (7), we use $cn < 1$ to derive

$$f(x) = f(0) + \frac{1}{n\mu}\left(1 - (1 - F(x))^n - cnF(x)\right),$$

$$> \frac{1}{n\mu}\left(1 - (1 - F(x))^n - F(x)\right),$$

$$= \frac{1}{n\mu}\left(1 - F(x)\right)\left(1 - (1 - F(x))^{n-1}\right).$$

The first line implies that the density diverges to infinity as $\mu \to 0$ if $F(x) = 1$, and the last line implies the same for $0 < F(x) < 1$. Since the density cannot diverge on an interval, $F(\cdot)$ has to be 0 on any open interval, so $F(x) = 0$ for $x < \bar{x}$.

Next, consider $c > 1/n$. We have to prove that $F(x) = 1$ for $x > 0$. Suppose not, and so $F(x) < 1$ for $x \in (x_a, x_b)$. From (7), we deduce $f(\bar{x}) = f(0) + (1 - cn)/n\mu$, which enables us to rewrite (7) as

$$f(x) = f(\bar{x}) + \frac{1}{n\mu}\left(cn(1 - F(x)) - (1 - F(x))^n\right),$$

$$> \frac{1}{n\mu}\left(1 - F(x)\right)\left(1 - (1 - F(x))^{n-1}\right).$$

The first line implies that the density diverges to infinity as $\mu \to 0$ if $F(x) = 0$ and $cn > 1$, and the last line implies the same for $0 < F(x) < 1$. Since the density cannot diverge on an interval, $F(\cdot)$ has to be 1 on any open interval, so $F(x) = 1$ for $x > 0$.

Finally, consider the case $c = 1/n$. In this case (7) becomes

$$f(x) = f(0) + \frac{1}{n\mu}\left(1 - F(x)\right)\left(1 - (1 - F(x))^{n-1}\right). \qquad (10)$$

This equation implies that the density diverges to infinity as $\mu \to 0$ when $F(x)$ is different from 0 or 1. Hence $F(\cdot)$ jumps from 0 to 1 at the mode M. Equation (10) implies that $f(0) = f(\bar{x})$, so the density is finite at the boundaries and the mode is an interior point. The location of the mode, M, can be obtained by rewriting (6) as $\mu f'/f = (1 - F)^{n-1} - 1/n$. Integrating both sides from 0 to $\bar{x}$, yields $\mu \ln(f(\bar{x})/f(0)) = M - \bar{x}/n$ (since $1 - F$ equals one to the left of M and zero to the right of M). The left side is zero since $f(0) = f(\bar{x})$, so $M = \bar{x}/n$. Q.E.D.

Although the above comparative static results may not seem surprising, they are interesting because they accord with economic intuition and patterns in laboratory data, but they are not predicted by a standard Nash equilibrium analysis. The results do not depend on auxiliary assumptions about the noise parameter μ (which is important because μ is not controlled in an experiment), but they only apply to steady-state situations in which behavior has stabilized.[11] For this reason we look to the last few rounds of experimental studies to confirm or reject logit equilibrium predictions. The data of Van Huyck, Battalio, and Beil (1990) indicate a huge shift in effort decisions (for a group size of 14-16 subjects) in experiments where c was zero as compared to experiments where c was ½. By the last round in the former case, almost all (96%) participants chose the highest possible effort, while in the latter case over three-quarters chose the lowest possible effort.[12] The numbers effect is also documented by quite extreme cases: the large group ($n = 14 - 16$) is compared to pairs of subjects (both for $c = ½$). They used both fixed and random matching protocols for the $n = 2$ treatment.[13] There was less dispersion in the data with fixed pairs, but in each case it is clear that effort decisions were higher with two players than with a large number of players.

Proposition 5 shows that with two players, $c = ½$ is a critical, knife-edge case that corresponds to the dividing line between all-top and all-bottom efforts in the limit as the noise vanishes. Goeree and Holt (1998) report experiments for two-person minimum-effort games for $c = .25$ and $c = .75$. Subjects were randomly matched for ten periods, and effort choices could be any *real* number on the interval [110, 170]. Initial decisions were uniformly distributed on [110, 170] for both the low-effort cost and high-effort cost treatments. However, average efforts increased in the low-cost treatment and decreased in the high-cost treatment. By the final period, the distributions of effort decisions were separated by the midpoint of the range of feasible choices, in line with Propositions 4 and 5. Moreover, with a noise parameter of $\mu = 8$ (estimated with data from a previous experiment, Capra *et al.*, 1999), equation (7) can be solved explicitly. The resulting logit equilibrium predictions for the average efforts were 127 for $c = .75$ and 153 for $c = .25$, with a standard deviation of 7 for both cases. These predicted averages are remarkably close to the data averages in the final three periods: 159 in the low-cost treatment and 126 in the high-cost treatment.[14]

It is important to point out that the techniques applied in this paper are not limited to the minimum-effort coordination game. Consider, for example, a three-person median-effort coordination game in which all three players receive the median, or middle, effort choice minus the cost of their own effort: $\pi_i(x_1,x_2,x_3) = \text{median}\{x_1,x_2,x_3\} - c\, x_i$, with c the effort-cost parameter.[15] This median-effort game has a continuum of asymmetric Pareto-ranked Nash equilibria in which two players choose a common effort level, x, and the third player chooses the lowest possible effort of zero. This asymmetric outcome is unlikely to be observed when players are randomly matched and drawn from the same pool, and it seems more sensible to characterize the entire population of players by a common distribution function F, with corresponding density f. The logit equilibrium condition is: $\mu f'(x) = \pi^{e\prime}(x) f(x)$. The marginal payoff function can be derived by noting that an increase in effort raises costs at a rate c and affects the median only if one of the other players is choosing a higher effort level and the other a lower effort level, which happens with probability $2F(1-F)$. Hence, the logit equilibrium condition becomes:

$$\mu f'(x) = f(x)\,(2\,F(x)\,(1 - F(x)) - c). \tag{11}$$

It is straightforward to show that the symmetric logit equilibrium for the median-effort game is unique and that an increase in cost results in a decrease of efforts.[16] Goeree and Holt (1998) report three sessions with this particular game form, with effort-cost parameters of $c = 0.1$, $c = 0.4$, and $c = 0.6$ respectively. The predictions for the final-period average effort levels that follow from (11) (with $\mu = 8$) are: 150 for $c = 0.1$, 140 for $c = 0.4$, and 130 for $c = 0.6$ with a standard deviation of 8 in each case. The observed average efforts in the last three periods for these sessions were 157, 136, and 113 respectively.[17]

As we show in the next section, the limiting Nash equilibrium determined in Proposition 5 corresponds to the one that maximizes the standard potential for the coordination game.

V. A Stochastic Potential for the Coordination Game

From the evolutionary game-theory literature it is well known that behavior in 2×2 coordination games converges to the risk-dominant equilibrium in the limit as noise goes to zero, as noted in the introduction. In 2×2 coordination games, the risk-dominant equilibrium can also be found by maximizing the potential of the game. Risk dominance is a concept that is difficult to apply to more general games, but there is a broad class of games for which there exists a potential function. Just as maximizing a potential function yields a Nash equilibrium, the introduction of noisy decision making suggests one might be able to use a stochastic potential function to characterize equilibria.[18] Here we propose such a stochastic potential function for continuous potential games.

Recall that a continuous *n*-person game is a potential game if there exists a function $V(x_1,...,x_n)$ such that $\partial V_i/\partial x_i = \partial \pi_i/\partial x_i$, $i = 1,..., n$, when these derivatives exist (see Monderer and Shapley, 1996). By construction, if a potential function, V, exists for a game, any Nash equilibrium of the game corresponds to a vector of efforts $(x_1,...,x_n)$ at which V is maximized in each coordinate direction.[19] Many coordination games are potential games. For instance, it is straightforward to show that the potential function for the minimum-effort coordination game is given by: $V = \min_{j=1,...,n}\{x_j\} - \Sigma^n_{i=1} c\, x_i$. Note that V includes the sum of all effort costs while the common effort is counted only once. Similarly, when payoffs are determined by the median effort minus the cost of a player's own effort, the potential function is the median minus the sum of all effort costs.

To incorporate randomness, we define a *stochastic potential*, which depends on the effort distributions of all players, as the expected value of the potential plus the standard measure of randomness, entropy. For the minimum-effort coordination game this yields:

$$V_S = \int_0^{\bar{x}} \prod_{i=1}^{n} (1 - F_i(x))\, dx - c \sum_{i=1}^{n} \int_0^{\bar{x}} (1 - F_i(x))\, dx - \mu \sum_{i=1}^{n} \int_0^{\bar{x}} f_i(x) \log(f_i(x))\, dx. \quad (12)$$

The first term on the right side is the expected value of the minimum of the *n* efforts, the second term is the sum of the expected effort costs, and the final term is the standard expression for entropy.[20] It is straightforward to show that the sum of the first two terms on the right side of (12) is maximized at a Nash equilibrium, whereas the final term is maximized by a uniform density, i.e. perfectly random behavior. Therefore, the noise parameter μ determines the relative weights of payoff incentives and noise. The following proposition relates the concept of stochastic potential to the logit equilibrium.

Proposition 6. The logit equilibrium distribution maximizes the stochastic potential in (12).

Proof. Anderson, Goeree, and Holt (1997) show that the stochastic potential is a Lyapunov function for an evolutionary adjustment model in which players tend to change their decisions in the direction of higher expected payoffs, but are subject to noise. The steady states of this evolutionary model correspond to the stationary points of the stochastic potential. The variational derivative of V_S with respect to F_i is:[21]

$$\frac{\partial V_S}{\partial F_i} \;=\; -\prod_{k\neq i}(1-F_k) \;+\; c \;+\; \mu\,\frac{f_i^{/}}{f_i}\,.$$

Equating this derivative to zero yields the logit equilibrium conditions in (5). Since the solution to this equation is unique (Proposition 3), and since V_S increases over time, the logit equilibrium is thus the unique maximum of the stochastic potential. Q.E.D.

When $\mu = 0$ the entropy term in (12) disappears, and the stochastic potential is simply the expected value of the potential $V = \min_{j=1,..,n}\{x_j\} - \Sigma^n_{i=1} c\, x_i$. Maximization requires that all players choose the same effort level, x, and the value of the potential then becomes: $x - n\, c\, x$. Hence, the potential is maximized at $x = \bar{x}$ if $c < 1/n$ and at $x = 0$ if $c > 1/n$, in accordance with Proposition 5. This link furnishes an explanation for Monderer and Shapley's (1996) claim that the potential function constitutes a useful selection mechanism. When $c = 1/n$ the ordinary potential refinement does not yield any selection since the potential is zero for any common effort level. In contrast, the stochastic potential *does* select a Nash equilibrium in the limit as the noise parameter goes to zero (at a common effort level of $\bar{x}/n$ - see Proposition 5). The critical value of c in Proposition 5 is similar to the condition that arises from applying risk dominance in 2×2 games. For example, if there are two effort levels, 1 and 2, and the payoff is the minimum effort minus the cost of one's own effort, the payoffs are given in Table I. This game has two pure-strategy Nash equilibria at common effort levels of 1 and 2.[22] Since this is a symmetric game, the risk dominant equilibrium is the best response to the other player choosing each decision with probability 1/2. Therefore, the high-effort outcome is risk dominant if $c < 1/2$, and the low effort outcome is risk dominant if $c > 1/2$.[23] The implication of

Proposition 5 with $n = 2$ is also that the lowest effort is selected in the limit if $c > 1/2$. For fixed c, Proposition 5 shows that the low effort equilibrium will be selected when there is a sufficiently large number of players.

The analysis of the limiting case as μ goes to zero is used only to show the relationship between the logit equilibrium, the potential refinement, and risk dominance; it is not intended to predict the actual behavior of players in a game. Indeed, our work is motivated by experiments in which noise is often pervasive. In many different types of experiments, behavior becomes less noisy after the first several periods as subjects gain experience. Nevertheless, dispersion in the data is often significant and stable in later periods. Moreover, the average decisions may converge to levels that are well away from the Nash prediction that results by letting μ go to zero.[24]

VI. Conclusion

Multiple equilibrium outcomes can result from externalities, e.g., when the productive activities of some individuals raise others' productivities. A coordination failure arises when these equilibrium outcomes are Pareto ranked. Coordination is more risky when costly, high-effort decisions may result in large losses if someone else's effort is low. This problem is not just a theoretical possibility; there is considerable experimental evidence that behavior in coordination games does not converge to the Pareto-dominant equilibrium, especially with large numbers of players and a high effort cost. Coordination problems have stimulated much theoretical work on selection criteria like evolutionary stability and risk dominance. Although coordination games constitute an important paradigm in theory, their usefulness in applications is limited by the need for consensus about how the degree of coordination is affected by the payoff incentives. Macroeconomists, for example, want to know how policies and incentives can improve the outcome (John, 1995).

This paper addresses the coordination problem that arises with multiple equilibria by introducing some noise into the decision making process. Whereas the maximization of the standard potential will yield a Nash equilibrium, we show that the logit equilibrium (introduced by McKelvey and Palfrey, 1995) can be derived from a stochastic potential, which is the expected value of the standard potential plus a standard entropic measure of dispersion. This logit equilibrium is a fixed point that can be interpreted as a stochastic version of the Nash equilibrium: decision distributions determine expected payoffs for each decision, which in turn determine the decision distributions via a logit probabilistic choice function. In a minimum-effort coordination game with a continuum of Pareto-ranked Nash outcomes, the introduction of even a small amount of noise results in a unique equilibrium distribution over effort choices. This equilibrium exhibits reasonable comparative statics properties: increases in the effort cost and in the number of players result in stochastically lower effort distributions, even though these parameter changes do not alter the range of pure-strategy Nash outcomes.

Despite the special nature of the minimum and median effort coordination games considered in this paper, the general approach should be useful in a wide class of economic models. Recall that a Nash equilibrium is built around payoff differences, however small, under the (correct) expectation that nobody else deviates. The magnitudes of payoff differences can matter in laboratory experiments, and changes in the payoff structure can push observed behavior in directions that are intuitively appealing, even though the Nash equilibrium of the game is unchanged. We have used the logit equilibrium elsewhere to explain a number of anomalous patterns in laboratory data.[25] We believe that this method of incorporating noise into the analysis of games provides an empirically based and theoretically constructive alternative to the standard Nash equilibrium analysis.

Appendix: Symmetric Mixed Strategy Nash Equilibria

In this Appendix we prove that the only symmetric, mixed-strategy Nash equilibria for the minimum effort game involve two-point distributions with unintuitive comparative static properties. Let $F^*(x)$ denote the common cumulative distribution of an individual's effort level, and let $G^*(x) = 1 - (1- F^*(x))^{n-1}$ be the distribution of the minimum of the other *n-1* effort levels. First consider the possibility that players randomize over a non-empty interval of efforts $[x_a, x_b]$ over which the common density, $f^*(x)$, is strictly positive. (This does not rule out atoms in the density outside of this interval.) Since a player's expected payoff must be constant at all effort levels played with positive density, the derivative of expected payoff in (4) derivative must be zero on $[x_a, x_b]$:

$$\pi^{e\prime}(x) \;=\; 1 - G^*(x) - c \;=\; (1 - F^*(x))^{n-1} - c \;=\; 0.$$

Thus $F^*(x)$ must be constant, contradicting the assumption that $f^*(x) > 0$ on this interval. Hence the density can only involve atoms.

We next show that there can be at most two atoms. Suppose instead there were three or more (distinct) such atoms: $x_a < x_b < x_c < ..$, etc. Let p_i denote the probability that the *minimum* of the other efforts is x_i, $i = a,b,c,...$ Then the expected payoffs for the three lowest of these efforts are:

$$\pi(x_a) \;=\; x_a - c x_a$$

$$\pi(x_b) \;=\; p_a x_a + (1 - p_a) x_b - c x_b$$

$$\pi(x_c) \;=\; p_a x_a + p_b x_b + (1 - p_a - p_b) x_c - c x_c$$

All effort levels played in a mixed strategy equilibrium entail the same profit level, and equating $\pi(x_b)$ to $\pi(x_c)$ yields: $(1 - p_a - p_b - c)(x_b - x_c) = 0$, and so $1 - p_a - p_b - c = 0$. This result then implies that $\pi(x_b) = \pi(x_c) = p_a x_a + p_b x_b$. Equating this expression to $\pi(x_a) = (1-c)x_a$ and using $1-p_a-p_b-c = 0$, we obtain $p_b x_a = p_b x_b$, a contradiction.

Finally, we must show that *any* two effort levels, $x_a < x_b$, can constitute a symmetric mixed strategy equilibrium. If all of the other players are using this mixed strategy, then it is never worthwhile to play $x > x_b$ (nothing can be gained), nor is it worthwhile to choose $x < x_a$ (reducing the minimum effort hurts because $c < 1$). Finally, for any $x_\lambda \equiv \lambda x_a + (1-\lambda)x_b$, with λ ε (0,1), the expected payoff is: $\pi(x_\lambda) = p_a x_a + (1-p_a)x_\lambda - cx_\lambda = \lambda\pi(x_a) + (1-\lambda)\pi(x_b)$. Thus it is never strictly better to choose an intermediate level(s) with positive probability since $\pi(x_a) = \pi(x_b)$.

Thus there is a continuum of mixed-strategy Nash equilibria. It is straightforward to show that they are Pareto-ranked like the pure-strategy Nash equilibria. Each of the mixed equilibria is characterized by a probability, q, of choosing x_b. Since there are n-1 other players, the expected payoff for this high effort choice is $(1 - q^{n-1})x_a + q^{n-1}x_b - cx_b$. Equating this expected payoff to the expected payoff for the low effort choice, $x_a - cx_a$, we obtain the equilibrium probability: $q = c^{1/(n-1)}$. Thus the probability of playing the higher effort level is *increasing* in the effort cost and the number of players. The intuition behind this result is that when effort costs go up, the way to make the players indifferent between high and low effort decisions is to raise the probability that a high effort decision will not be in vain, i.e. to raise the probability of high effort. Similarly, increasing the number of players while maintaining equality of expected payoffs requires a constant probability that the minimum effort is low. This implies that the probability that a single player chooses the low effort is reduced as n is increased. To summarize, the mixed-strategy Nash equilibria consist of a set of Pareto-ranked, two-point distributions with unintuitive comparative statics.

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		Player 2's Effort	
		1	2
Player 1's Effort	1	1 - *c*, 1 - *c*	1 - *c*, 1 - 2*c*
	2	1 - 2*c*, 1 - *c*	2 - 2*c*, 2 - 2*c*

Figure 1. A 2×2 Coordination Game

1. The literature on coordination game experiments is surveyed in Ochs (1995).

2. The condition on payoff parameters that determines the limiting effort levels reflects the risk-dominance condition for 2×2 games, and is analogous to the limit results of Foster and Young (1990), Young (1993), and Kandori, Mailath, and Rob (1993) for evolutionary models.

3. This is sometimes called a "stag-hunt" game. The story is that a stag encircled by hunters will try to escape through the sector guarded by the hunter exerting the least effort. Thus the probability of killing the stag is proportional to the minimum effort exerted.

4. When the additive preference shocks for each possible decision are independent and double-exponential, then the logit equilibrium corresponds to a Bayes/Nash equilibrium in which each player knows the player's own vector of shocks and the distributions from which others' shocks are drawn. Alternatively, the logit form can be derived from certain basic axioms. Most important is an axiom that implies an independence-of-irrelevant-alternatives property: that the ratio of the choice probabilities for any two decisions is independent of the payoffs associated with any other decision (see Luce, 1959). This property, together with the assumption that adding a constant to all payoffs will not affect choice probabilities, results in the exponential form of the logit model.

5. McKelvey and Palfrey (1995) use the logit form extensively, although they prove existence of a more general class of quantal response equilibria for games with a finite number of strategies. It can be shown that the quantal response model used by Rosenthal (1989) is based on a linear probability model. Chen, Friedman, and Thisse (1997) use a probabilistic choice rule that is based on the work of Luce (1959).

6. This distribution has numerous applications in biology and epidemiology. For example, the logistic function is used to model the interaction of two populations that have proportions $F(x)$ and $1 - F(x)$. If $F(x)$ is initially close to 0 for low values of x (time), then $f(x)/F(x)$ is approximately constant, and the growth (infection rate) in the proportion $F(x)$ is approximately exponential (see, e.g., Sydsaeter and Hammond, 1995). Visually, the logistic density has the classic "normal" shape.

7. Proposition 3 below shows that the truncated logistic in (8) is the only solution for $n = 2$.

8. The n-player solution for $\bar{x} = \infty$ was obtained by observing that $F(x)$ is the distribution of the minimum of the other player's effort when $n = 2$. In general, the minimum of the $n - 1$ other efforts is $G(x) = 1 - (1 - F(x))^{n-1}$. The solution was found by conjecturing that $G(x)$ is a generalized logistic function, and then determining what the constants have to be to satisfy the equilibrium condition (7) and the boundary conditions, $F(0) = 0$ and $F(\infty) = 1$. This procedure yields the symmetric logit equilibrium for the case of $\bar{x} = \infty$ and $c > 1/n$ as the solution to

$$1 - (1 - F(x))^{n-1} = \frac{nc(nc-1)}{nc - 1 + \exp(-(n-1)cx/\mu)} - (nc-1). \qquad (*)$$

The proof (which is available from the authors on request) is obtained by differentiating both sides of (*) to show that the resulting equation is equivalent to (7). Notice that (*) satisfies the boundary conditions, and that the left side becomes $F(x)$ when $n = 2$. The solution in (*) is relevant if $nc - 1 > 0$. It is straightforward (but tedious) to verify that the equilibrium effort distribution in (*) is stochastically decreasing in c and n, and increasing in μ (for $c > 1/n$), as shown in Proposition 4 below.

9. For example, we can consider the generalization of (1): $\pi_i = (\sum_j x_j^{\rho})^{1/\rho} - cx_i$, and then study the limit behavior of the unique equilibrium as $\rho \rightarrow \infty$, which is the Leontief limit of the CES function as given in the text.

10. McKelvey and Palfrey (1995) show that the limit equilibrium as μ goes to zero is always a Nash equilibrium for finite games, but that not all Nash equilibria can be necessarily found in this manner. Proposition 5 illustrates these properties for the present continuous game.

11. McKelvey and Palfrey (1995) estimated μ for a number of finite games, and found that it tends to decline over successive periods. However, this estimation applies an *equilibrium* model to a system that is likely adjusting over time. Indeed, the decline in estimated values of μ need not imply that error rates are actually decreasing, since behavior normally tends to show less dispersion as subjects seek better responses to others' decisions. This behavior is consistent with results of Anderson, Goeree, and Holt (1997) who consider a dynamic adjustment model in which players change their decisions in the direction of higher payoff, but subject to some randomness. They show that when the initial data are relatively dispersed, the dispersion decreases as decisions converge to the logit equilibrium. This reduction would result in a decreasing sequence of estimates of μ, even though the intrinsic noise rate is constant.

12. The numbers reported are 72% for one treatment, and 84% for another. The second treatment (their case *A*') differed from the first in that it was a repetition of the first (although with five rounds instead of ten) that followed a $c = 0$ treatment. The fact that there were more lowest-level decisions after the second treatment (when subjects were even more experienced) may belie our taking the last round in each stage to be the steady-state - although the difference is not great.

13. Half of the two-player treatments were done with fixed pairs, and the other half were done with random rematching of players after each period. Of the 28 final-period decisions in the fixed-pairs treatment, 25 were at the highest effort and only 2 were at the lowest effort. The decisions in the treatment with random matchings were more variable. The equilibrium model presented below does not explain why variability is higher with random matchings. Presumably, fixed matchings facilitate coordination since the history of play with the same person provides better information about what to expect. Another interesting feature of the data that cannot be explained by our equilibrium model is the apparent correlation between effort levels in the initial period and those in the final period in the fixed-pairs treatments.

14. The model used here is an equilibrium formulation that pertains to the last few rounds of experiments, when the distributions of decisions have stabilized. An alternative to the equilibrium approach taken here is to postulate a dynamic adjustment model. For instance, Crawford (1991, 1995) presents a model in which each player in a coordination game chooses effort decisions that are a weighted average of the player's own previous decision and the best response to the minimum of previous effort choices (including the player's own choice). This partial adjustment rule is modified by adding individual-specific constant terms and independent random disturbances. This model provides a good explanation of dynamic patterns, but it cannot explain the effects of effort costs since these costs do not enter explicitly in the model (the best response to the minimum of the previous choices is independent of the cost parameter).

15. In Van Huyck, Battalio, and Beil's (1991) median-effort game players also receive the median of all efforts but a cost is added that is quadratic in the distance between a player's effort and the median effort. The latter change may have an effect on behavior and could be part of the reason why the data show strong "history dependence."

16. Equation (11) can be integrated as: $\mu f(x) = f(0) + F(x)^2 - 2/3\, F(x)^3 - c\, F(x)$. The proof that the solution to this equation is unique is analogous to the proof of Proposition 3. The proof that an increase in c leads to a decrease in efforts is analogous to the proof of Proposition 4.

17. Notice that two of the three averages are within one standard deviation of the relevant theoretical prediction.

18. Indeed, Young (1993) has introduced a different notion of a stochastic potential for finite, n-person games. He shows that the stochastically stable outcomes of an evolutionary model can be derived from the stochastic potential function he proposes.

19. Note, however, that V itself is not necessarily even locally maximized at a Nash equilibrium, and, conversely, a local maximum of V does not necessarily correspond to a Nash equilibrium.

20. It follows from partial integration that the expected value of player i's effort is the integral of $1-F_i$, which explains why the second term on the right side of (10) is the sum of expected effort costs. To interpret the first term, recall that the distribution function of the minimum effort is $1 - \prod^n_{i=1} (1-F_i)$, and therefore the expected value of the minimum effort is the integral of $\prod^n_{i=1} (1-F_i)$. The third term (including the minus sign) is a measure of randomness that is maximized by a uniform density.

21. Recall that the variational derivative of $\int I(F, f)\, dx$ is given by $\partial I/\partial F - d/dx\, (\partial I/\partial f)$.

22. There is also a symmetric mixed strategy equilibrium, which involves each player choosing the low effort with probability $1-c$. This equilibrium is unintuitive in the sense that a higher effort cost *reduces* the probability that the low effort level is selected.

23. Straub (1995) has shown that risk dominance has some predictive power in organizing data from 2×2 coordination games in which players are matched with a series of different partners.

24. For instance, in a "traveler's dilemma" game, there is a unique Nash equilibrium at the lowest possible decision, and this equilibrium would be selected by letting μ go to zero in a logit equilibrium. For some parameterizations of the game, however, observed behavior is concentrated at levels slightly below the highest possible decision, as is predicted by a logit equilibrium with a non-negligible noise parameter (Capra, Goeree, Gomez, and Holt, 1999). Thus the effects of adding noise in an *equilibrium* analysis may be quite different from starting with a Nash equilibrium and adding noise around that prediction. The traveler's dilemma is an example where the equilibrium effects of noise can "snowball," pushing the decisions away from the unique Nash equilibrium to the opposite side of the range of feasible decisions.

25. In rent-seeking contests where the Nash equilibrium predicts full rent dissipation, the logit equilibrium predicts that the extent of dissipation will depend on the number of contestants and the cost of lobbying effort (Anderson, Goeree, and Holt, 1998). Moreover, the logit equilibrium predicts over-dissipation for some parameter values, as observed in laboratory experiments. The effect of endogenous decision error is quite different from adding symmetric, exogenous noise to the Nash equilibrium. This is apparent in certain parameterizations of a "traveler's dilemma," for which logit predictions and laboratory data are located near the highest possible decision, whereas the unique Nash equilibrium involves the lowest possible one (Capra, Goeree, Gomez, and Holt, 1999).