The War of Attrition with Noisy Players

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ABSTRACT

We first consider the Nash equilibria for the two-player normal-form war of attrition, which is equivalent to a second-price all-pay auction. When there is a limit to the maximum effort (for example, a budget constraint), and for low enough prize values, there exists a symmetric mixed-strategy equilibrium under equal prize values. However, the only equilibria when prize values differ involve one player choosing zero effort (conceding immediately). Non-degenerate mixed-strategy equilibria under different prize values reemerge when there is no maximum effort. These equilibria have perverse comparative static properties: an increase in one player's value leaves that player's bid distribution unaffected and raises the other player's bid (in the sense of first-degree stochastic domination). Given these quirks of the Nash equilibrium, we then describe the (boundedly-rational) logit equilibrium for the game. This equilibrium is unique and symmetric when players have the same prize values. Moreover, the player with the higher prize value exerts more effort (in the sense of first-degree stochastic dominance). The paper concludes with a discussion of some experimental evidence for a related game, that supports the qualitative predictions of the logit equilibrium model.

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I. Introduction

When the outcome of competition for a prize is determined by who pays the higher price (e.g. bidding on a prize in a standard first-price auction), only the winner has to pay. But there are many situations in which some kind of real effort is the deciding factor, e.g. head-butting in animal conflict (Maynard Smith, 1974, Maynard Smith and Parker, 1976, and Riley, 1980). This kind of conflict is sometimes called a "war of attrition" since the competition may turn out to be quite costly for all concerned. In a war of attrition, the successful competitor needs only to top the competitor's effort and cease just after the other quits. In this sense, the war of attrition is similar to a second-price auction in which the winning (high) bidder pays only the bid of the second highest bidder. The difference is that all effort costs must be incurred in a war of attrition, whereas the losing bidders need not pay anything in a second-price auction. Thus the war of attrition is formally equivalent to a second-price all-pay auction. Besides animal conflict, the war of attrition has been used to model political contests (Bulow and Klemperer, 1997), standard-setting games (Farrell, 1996), strikes (Kennan and Wilson, 1989) and firm exit from a declining industry (Ghemawat and Nalebuff, 1985, and Fudenberg and Tirole, 1986). The war of attrition also has a strong similarity to the economic analysis of patent races (see e.g. Lee and Wilde, 1980).

The war of attrition, when analyzed as a normal-form game, has the interesting property that the Nash equilibria in pure strategies are asymmetric, even when players value the prize equally. In the two-player case, it is an equilibrium for one player to select zero effort (concede) and for the other player to select an effort level with an associated cost that exceeds the value of the prize. This aggressive strategy turns out to be costless when the rival concedes immediately, because the cost incurred by the winner is determined by the effort exerted by the loser. Moreover, the player who concedes cannot gain by deviating to an effort level that will cost more than the prize value. Moulin (1981, p.181) notes that this strong asymmetry "lacks realism" in a symmetric game where neither player has the means to signal a commitment to a high effort. There exist mixed-strategy equilibria in which both players exert non-zero effort. However, these mixed equilibria exhibit counter-intuitive comparative static properties. For instance, an increase in a player's prize value leaves that player's bid unaltered, but lowers the other player's bid (in the sense of first-degree stochastic dominance).

This paper then reconsiders the problem of equilibrium behavior in a normal-form war-of-attrition game with two players, allowing for the possibility that players are imperfectly rational. Following Rosenthal (1989), McKelvey and Palfrey (1995), and Chen, Friedman, and Thisse (1996), we relax the assumption that players always choose the decision with the highest payoff. Instead, we assume that choice probabilities are positively but imperfectly related to expected payoffs via a logit probabilistic choice rule. The logit choice probabilities for each player are specified to be exponential functions of expected payoffs, which in turn are determined by the choice probabilities of other players. The equilibrium condition is that the expected payoffs used to calculate the logit choice probabilities are equal to those determined by the choice probabilities of the other players. This approach, therefore, imposes a Nash-like consistency-of-actions-and-beliefs condition, with the difference from a standard Nash analysis being that actions are probabilistic functions of payoffs. The errors determined by the probabilistic choice specification are determined by an error parameter, μ, which captures perfect rationality in the limit as μ goes to zero, and perfectly random (payoff-independent) behavior as μ goes to infinity. Thus the approach can be thought of as a generalization of the Nash equilibrium to allow for noise in decision making. It is important to stress that errors are not simply added ex post to the Nash equilibrium. Rather, the model is an *equilibrium* analysis of bounded rationality.

Building on the existence proof in Anderson, Goeree, and Holt (1997c) for the case of a first-price all-pay auction, we show that a logit equilibrium for the war of attrition game exists, whether or not players' prize values are the same. With symmetric values, the logit equilibrium is unique and symmetric across players. Moreover, an increase in the common prize value leads to stochastically higher efforts. When players' prize values differ, the player with a higher prize value exerts more effort, which is not the case in the mixed-strategy Nash equilibrium. All of these results follow from a simple graphical analysis of the logit distribution functions of the two players. This method of analysis should prove useful in other applications of the logit equilibrium.

The next section presents the notation and assumptions, and gives a derivation of the expected payoff for the normal-form war of attrition. In section III we analyze the Nash equilibria in detail: we derive the pure-strategy and mixed-strategy Nash equilibria for the case of a finite and an infinite maximum possible effort. In section IV we consider a bounded rationality equilibrium for the war of attrition, and derive some of its properties. The final section concludes.

II. The Model

In the normal-form representation of the two-player war of attrition, each player chooses an effort level, e_i, to compete for a single prize worth v_i, $i = 1,2$. The cost of a unit of effort is c, the same for both players. Feasible effort choices range from 0 to $\bar{e}$, where the upper bound $\bar{e}$ may be infinite. A finite upper bound has been interpreted as a budget constraint (see Che and Gale, 1998). The winner is the player who exerts the greater effort, and the winner's payoff is the winner's prize value, v_i, minus the cost of the *other* player's effort, ce_{-i}. The loser receives no prize, but does have to pay the cost of the effort exerted.[1] This means that the normal-form war of attrition is equivalent to a second-price all-pay auction.

In order to find the Nash equilibria of the normal-form war of attrition (as well as the logit equilibrium analyzed in section IV) we need the expected payoff, $\pi_i^e(e)$, that player i obtains from choosing an effort level e. This expected payoff depends both on the effort exerted and on the effort distribution function of the other player, which we denote by $F_{-i}(e)$ with corresponding density $f_{-i}(e)$. Player i's expected payoff consists of two terms, corresponding to whether or not player i's effort is the higher effort:

$$\pi_i^e(e) = \int_0^e (v_i - cy) f_{-i}(y)\, dy - ce(1 - F_{-i}(e)), \quad i = 1,2. \tag{1}$$

The first term on the right side applies when player i wins the prize and has to pay the expected cost of the other player's effort, and the second term is player i's own cost, ce, times the probability of losing, $(1 - F_{-i})$. The derivative of expected payoff with respect to a player's own effort is therefore given by

$$\pi_i^{e\prime}(e) = v_i f_{-i}(e) - c(1 - F_{-i}(e)), \quad i = 1,2. \tag{2}$$

Thus the profit derivative is the value of the marginal increase in the probability of winning

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[1] If there is a tie, players have a 50 percent chance of winning.

minus the extra effort cost incurred when the player loses the contest. The slope of expected payoff in (2) will be important for the analysis of both the Nash and logit equilibria. In a mixed-strategy equilibrium a player should be indifferent over the actions taken, i.e. (2) should be zero at any action taken with positive probability density, and in a logit equilibrium the slope of a player's equilibrium density will be proportional to the slope in (2).

III. Nash Equilibria

In this section we derive the Nash equilibria of the normal-form war of attrition. We suppose throughout that $v_1 \geq v_2$. The characterization of equilibrium is crucially dependent on the prize value in relation to the maximum possible effort cost, $c\bar{e}$. We first show in part (a) that the only equilibrium involves both players choosing the maximum possible effort if prize values are high enough. For lower prize values, the only pure strategy equilibria are asymmetric, with one player choosing zero effort. This case is covered in part (b). In part (c) we consider the symmetric mixed-strategy equilibrium when prize values are the same for both players. Finally, in part (d) we consider the possibility of other types of equilibria for low prize values. Surprisingly, when prize values are equal, there exists a symmetric mixed-strategy equilibrium that cannot arise as the limit of an asymmetric mixed-strategy equilibrium.

(a) High Prize Values.

When $v_i/2 > c\bar{e}$, $i = 1,2$, it is a dominant strategy for each player to choose the maximum effort, $\bar{e}$, winning for sure when the other player chooses an effort level below $\bar{e}$, and winning with probability one half when the other player also chooses $\bar{e}$. To see that this is an equilibrium, just note that playing $\bar{e}$ guarantees a win if the rival plays below $\bar{e}$ and winning garners v_i at the rival's effort cost, and $v_i > ce$ for all $e \in [0, \bar{e}]$; furthermore, when the rival plays $\bar{e}$ it is better to match $\bar{e}$ than to play zero since the expected rent is positive. Thus the only equilibrium is to play $\bar{e}$.

Both players choosing an effort level of $\bar{e}$ is still an equilibrium when $v_i/2 = c\bar{e}$ (for the same reasons as above, except that playing $\bar{e}$ is only a weakly dominant strategy for player 2). But another type of equilibrium also appears, providing the segue to the next case. This equilibrium involves player 2 choosing a zero effort, while player 1 chooses an effort level equal to $\bar{e}$. Player 2 can do no better by choosing $\bar{e}$ (rents are zero in both cases) and does worse by choosing any action in $(0, \bar{e})$ with positive probability since such actions lose for sure but incur effort costs, and thus entail negative profits. Likewise, player 1 does no better with a lower effort, and lower efforts cannot be part of an equilibrium strategy since player 2's best reply would be to choose $\bar{e}$.

<u>(b) Pure-strategy Equilibria for Lower Prize Values.</u>

Suppose first that $v_1 \geq 2c\bar{e}$ but that $v_2 < 2c\bar{e}$. It is a (weakly) dominant strategy for player 1 to play the maximum possible effort, $\bar{e}$, for the same reasons as above, but then player 2 must choose zero effort. This is the unique equilibrium if $v_2 \geq 2c\bar{e}$ since player 1 must play $\bar{e}$ for player 2 not to wish to top player 1's effort. But when $v_2 < 2c\bar{e}$, player 1 can choose any effort between v_2/c and $\bar{e}$ without player 2 being able to profitably top 1's choice. However, the equilibrium remains fundamentally unique in that the low value player chooses zero effort, while the high value player attains full rent.

This brings us to the case $v_1/2 < c\bar{e}$, which is the interesting one because then the effort limit no longer is a binding constraint in the sense that both playing $\bar{e}$ yields negative payoffs to both: players have room to manoeuvre. The only pure-strategy equilibria in this case are again asymmetric, but no longer fundamentally unique. They involve one player choosing zero effort while the other player's effort lies between v_{-i}/c and $\bar{e}$ (or $\bar{e}$ if this interval does not exist); the high effort player gets the prize for sure at no cost while the low-effort player would lose by topping the higher effort. Of course, there is a continuum of equilibria in which the high-effort player chooses a distribution with support $[v_{-i}/c, \bar{e}]$ with the lower-effort player choosing zero, but the fundamental property of such equilibria is that one player earns zero choosing zero effort while the other earns a rent of v.[2] Such payoffs asymmetries are perhaps unappealing in a symmetric game, when values are the same, and lead one to consider non-degenerate mixed-strategy equilibria.

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[2] There are also other mixed-strategy equilibria with wider support that satisfy the no-overtaking condition but still with the fundamental property that one player chooses zero effort.

<u>(c) Symmetric Mixed-Strategy Equilibria.</u>

When values are the same for both players, $v_1 = v_2 = v$, then there exists a symmetric equilibrium. If $v/2 \geq c\bar{e}$, the symmetric equilibrium involves both players choosing $\bar{e}$, as per part (a). If $v/2 < c\bar{e}$, the equilibrium involves non-degenerate mixed strategies. Any such equilibrium has the following properties:

(i) there is no atom except possibly at $\bar{e}$;

(ii) the support begins at zero;

(iii) the support is a contiguous interval $[0, \hat{e}]$, where $\hat{e} \leq \bar{e}$.

The first property follows since a lower atom would be profitably topped by a player with infinitesimal increase in effort costs when beaten, and a finite increase in the probability of winning. The second property holds because otherwise profits would be negative: playing some lowest effort above zero would never win the prize but entails a positive effort cost. The third property (no gaps up to $\hat{e}$) follows from (ii) and the fact that moving probability mass from the top of any purported interval to the bottom involves no decrease in the win probability but a positive decrease in effort costs in case the player loses.

Thus any symmetric mixed-strategy equilibrium involves both players mixing over $[0, \hat{e}]$, with possibly a spike at $\bar{e}$. On $[0, \hat{e}]$, payoffs have to be constant, and so (from (2))

$$\pi^{e\prime}(e) = v\,f(e) - c\,(1 - F(e)) = 0, \tag{3}$$

or

$$\frac{f(e)}{1 - F(e)} = \frac{v}{c}. \tag{4}$$

Since the left side is minus the derivative of $\ln(1 - F)$, the solution is $1 - F = k\exp(-ce/v)$, where the constant of integration, k, is determined by the boundary condition $F(0) = 0$ (by (i)) as $k = 1$. It is readily verified that when the maximum possible effort, $\bar{e}$, is infinite, the solution is simply $F = 1 - \exp(-ce/v)$ for $e \in [0, \infty]$ (see for instance Moulin, 1981).

When $\bar{e}$ is finite, but still $\bar{e} > v/2c$, then the equilibrium consists of a density on $[0, \hat{e}]$ (given by (4)) and an atom at $\bar{e}$, which is such that choosing $\bar{e}$ also yields zero profit. Since $\pi^e(0) = 0$ and $\pi^{e\prime}(0) = 0$ on $[0, \hat{e}]$, this is an equilibrium because any effort choice in $(\hat{e}, \bar{e})$

entails no higher probability of winning than an effort $\hat{e}$, but it entails a higher cost than at $\hat{e}$ when the rival plays $\bar{e}$. Thus the proposed equilibrium satisfies $F = 1 - \exp(-ce/v)$ for $e \in [0, \hat{e}]$ and an atom of size $S = \exp(-c\hat{e}/v)$ at $\bar{e}$. To determine the value of $\hat{e}$, consider what happens when a player increases effort from $\hat{e}$ to $\bar{e}$: the expected gain is $Sv/2$ (when the player ties with the rival at $\bar{e}$) while the increase in effort costs is $S(c\bar{e} - c\hat{e})$. So, in order for a player to be indifferent between $\hat{e}$ and $\bar{e}$, we need $\hat{e} = \bar{e} - v/2c$.

(d) Asymmetric Mixed-Strategy Equilibria.

One would perhaps expect that the symmetric mixed-strategy equilibrium can be obtained as the limit of the mixed-strategy equilibrium with asymmetric values. As we now show, this is not the case. We proceed by characterizing the properties that any non-degenerate mixed-strategy equilibrium must satisfy, and then show that these lead to a contradiction unless either both values are equal (as per (c) above) or else the maximum possible effort, $\bar{e}$, is infinite.

First note that it is not possible that the first effort level played with positive probability density by either player exceeds zero. If the player's purported equilibrium strategies were to start with atoms at the same point, each player would be better off moving the atom to a slightly higher effort level. If one atom preceded another, or indeed if one player's starting point preceded the other's, the lower player reduces effort costs without changing winning probabilities by shifting all such preceding mass back to zero. The same argument holds (for the atom player), if one player starts with an atom while the other starts with a density at the same point. The last such case involves both starting with densities. But if this were really part of an equilibrium strategy, expected payoffs would be negative (the winning probability at such a point would be zero, but at positive effort cost) and zero payoff can be guaranteed by choosing zero effort.

So suppose that one player has an atom of less than full mass at zero (below we allow for such an atom to have zero mass). It is not possible for both players to have atoms at zero effort, since either could just top the other. Nor is it possible that the atom is of full mass, otherwise we have case (b) above. So the other player must start at some effort level before the first one is played out (if not, the first is better off with all mass at zero effort). Moreover, the other player must also start at zero, as we next show. Indeed, suppose not. Then the argument of the preceding paragraph applies to the continuation game, so there can be no gap above zero effort before players resume.

Next, there can be no atom strictly before one of the players is played out (i.e., before the highest strategy played - note that this condition does allow for a final atom). If one player's strategy involved such an atom, then the other player would necessarily play with positive density just before the atom or else the atom would be moved down (which would reduce effort costs without affecting win probabilities). But then the other player would get strictly larger payoff from moving probability density from below the atom to above it (resulting in a finite rise in win probability with infinitesimal increase in effort cost).

Likewise, we can show that there can be no gaps in the support of the purported equilibrium density strictly before a player is played out. If one player's density exhibits a gap while the other's does not, then the player without a gap can move density from the gap to its lower end and decrease effort cost without changing the probability of winning. Similarly, if both players had gaps over the same interval, they cannot be indifferent playing both ends of the gap since the upper end involves the same win probability as the lower end but strictly higher effort costs.

Thus both players must play with positive density from zero up to some level $\hat{e}$, and one player might have an atom at zero. Now, it is not possible that both players have atoms above $\hat{e}$, except possibly at $\bar{e}$, which cannot be topped. So we can characterize equilibrium distributions under the conditions tied down above.

Denote player A as the one with an atom (possibly zero) at zero. Then on $[0, \hat{e}]$, the candidate distributions must satisfy

$$1 - F_i = k_i \exp(-ce/v_i), \qquad i = A, B,$$

(see the discussion following equation (4)). Hence $k_B = 1$ since $F_B(0) = 0$, and $k_A = 1 - F_A(0)$. But then both distributions are less than unity at any finite $\hat{e}$. This means that if $\bar{e}$ is finite (the infinite case is picked up below) then each player must have an atom at $\bar{e}$. But then, from the condition at the end of part (c) applied to the asymmetric case we have $S_i(c\bar{e} - c\hat{e} - v_{-i}) = 0$, where S_i is the atom size of player i. Unless values are identical, this is inconsistent with both players choosing positive atoms, so we have a contradiction.

Thus the symmetric mixed-strategy equilibrium described in (c) is very brittle - there is no such equilibrium type for asymmetric values unless $\bar{e}$ is infinite. In the latter case the support is the real line and there can be no atoms (except possibly at zero for one player). The equilibrium distributions without atoms are given from (5) with $k_i = 1$ by[3]

$$F_i = 1 - \exp(-ce/v_{-i}).$$

The interesting feature of these distributions is that the player with the higher value has the higher distribution ($F_1 > F_2$ as $v_1 > v_2$), which follows from the indifference property of the mixed-strategy equilibrium that each player's equilibrium mixed-strategy must render the other player indifferent. These results imply that the player with the higher value chooses a stochastically lower effort, which is counter to one's basic economic intuition that if the prize is worth more then you should try harder to get it. Another counter-intuitive facet of the equilibrium concerns its comparative static properties. In particular, raising the prize value of one player leaves unaffected that player's equilibrium distribution, but *lowers* the distribution of the other player. This makes sense in terms of the mixed-strategy: the rival must try harder in order to retain the indifference of the player whose prize value has increased, but casual intuition would suggest that a higher value should make the player herself try harder, with the other player cutting back as a consequence.

IV. The Logit Equilibrium

The multiplicity of Nash equilibria is a worrisome feature because it entails a lack of predictive power. Moreover, as we have shown, some Nash equilibria involve counter-intuitive comparative static properties, that also run against some experimental findings for similar games (see the discussion in the conclusions). These results suggest a need for models of behavior that can generate more robust conclusions. One avenue that is inspired from the literature on bounded rationality as well as experimental evidence is that players do not always choose the best action.

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[3] The equilibria with one player having an atom at zero (of less than full mass) involve that player using the distribution $F_i(e)=1 - (1 - F_i(0)) \exp(-ce/v_i)$ for any $F_i(0) \in (0,1)$, with the other player using the distribution given in the text.

Therefore, we shall assume that players behavior is "noisy," i.e. there is some chance that players make non-optimal decisions, or errors, with the probability of a such an error being inversely related to its severity.

There are several plausible interpretations of the noise (Anderson, Goeree, and Holt, 1997d). A bounded-rationality interpretation is that individuals make mistakes in evaluating the profitability of alternative strategies, but that more costly mistakes are less common. This feature is captured by assuming that choice probabilities are positively related to expected payoffs. A second interpretation of the "noise" is perhaps preferred by most economists: individuals are rational, but there are unmeasured or unobserved non-monetary aspects of the strategies. When the difference between the expected payoffs of two decision is large, then this difference should dominate all but the most extreme deviations from measured preferences, but noise will become more important when payoff differences are small. Which story is better may depend on the specific application, but the source of the randomness has no effect on the way that we model it.

We incorporate errors by specifying a player's effort density to be an increasing function of expected payoff, but without necessarily having all of probability mass located at the decision that maximizes expected payoff. The particular parametric form that we will assume is the logit rule which specifies decision probabilities to be proportional to an exponential function of expected payoffs:

$$f_i(e) \;=\; k_i \exp(\pi_i^e(e)/\mu), \quad i = 1, 2, \tag{5}$$

where k_i is a constant that ensures that the density integrates to one and μ is an error parameter. Since the choice probabilities are proportional to exponential functions of expected payoffs, so the *ratio* of choice probabilities for two decisions are functions of the *difference* between the expected payoffs for those decisions. Thus when payoff differences are large, mistakes are less likely. More precisely, when μ tends to zero, the decision with the highest expected payoff gets more and more weight, whereas all decisions become equally likely (complete randomness) when μ tends to infinity. Note from (1) that the expected payoff in (3) is zero at e = 0 (zero effort implies that the prize is never won), so $k_i = f_i(0)$.[4] Since the expected payoff of player *i* depends on the distribution of the other player, the equations in (5) do not provide closed-form solutions for the logit density; instead they constitute fixed-point equations which the logit densities have to solve. To get some insight into the properties of these solutions it will prove useful to differentiate (5) with respect to the effort level *e*, which yields: $f_i'(e) = f_i(e)\,\pi_i^{e\prime}(e)/\mu$. Substituting the expression for expected payoff from (1), we obtain:

$$\mu\, f_i^{\prime}(e) \;=\; f_i(e)\left(v_i\, f_{-i}(e) \;-\; c\,(1 - F_{-i}(e))\right), \qquad i \;=\; 1,2. \tag{6}$$

Equation (6) is used below to establish some general properties of the logit equilibrium. Specifically, we prove symmetry and uniqueness of the equilibrium when prize values are equal, we determine the effect of different prize values, and we derive the comparative static effects of changes in values and costs. We start out with the existence of a logit equilibrium.

Proposition 1. There exists a logit equilibrium for the normal-form war of attrition. Moreover, the logit equilibrium densities are infinitely differentiable.

Proof. The existence proof is similar to the one in Anderson, Goeree, and Holt (1997c), which deals with the existence of a logit equilibrium for a first-price all-pay auction. The method of proof is to define player *i*'s logit equilibrium density as a fixed-point of an operator, *T*, that is continuous in *i*'s effort and in the other player's distribution function. Since this operator is defined on a non-compact function space, one cannot simply rely on Brouwer's theorem to prove the existence of a fixed-point. However, by showing that *T* is a compact operator, one can use Schauder's extension of Brouwer's theorem to prove existence of a logit equilibrium.

Differentiability of the logit equilibrium densities follows from the definition in (5) together with continuity of the expected payoff function in the rival's distribution. Note that by

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[4] Since the expected payoff in (1) is finite for all possible effort choices, the logit density is finite for all non-zero values of μ. Therefore, the resulting distribution functions are continuous, and the probability of ties is zero in a logit equilibrium.

partially integrating the first term on the right of (2), the expected payoff can be written as

$$\pi_i^e(e) = v_i F_{-i}(e) - c\int_0^e (1 - F_{-i}(y))dy, \quad i = 1,2.$$

Since the logit equilibrium distributions are always continuous (see footnote 3), the expected payoff will be continuous, and since the density is just an exponential function of expected payoff, also the density is continuous. But this implies that the distribution function is differentiable, and thus so are the expected payoff and the logit density. Clearly, this line of reasoning can be repeated *ad infinitum*, proving that the logit densities are "smooth," i.e. infinitely differentiable. Q.E.D.

The next issue is symmetry. Proposition 2 shows that the logit equilibrium is symmetric when players' prize values are equal. When values differ, the player with the higher value will choose stochastically higher efforts.

Proposition 2. In a logit equilibrium for the symmetric normal-form war of attrition, players have identical bid distributions. When players' values differ, the player with the higher value exerts more effort (in the sense of first-degree stochastic dominance).

Proof. We start by proving the final statement of the proposition. Let F_1 and F_2 denote the distributions corresponding to v_1 and v_2, where $v_1 > v_2$. We have to prove that $F_1(e)$ lies below $F_2(e)$ for all $0 < e < \bar{e}$. Suppose, in contradiction, that $F_1 > F_2$ for some effort levels, as shown in Figure 1. Any region of divergence between the distribution functions will have a maximum *horizontal* difference, as indicated by the horizontal dashed line at height $F^* = F_1(e_1) = F_2(e_2)$. The first-order condition for the distance to be maximized at F^* is that the slopes of the distribution functions be identical at F^*, i.e. $f_1(e_1) = f_2(e_2)$. The second-order condition is that the slope of F_1 increases no slower than F_2, i.e. $f_2'(e_2) \geq f_1'(e_1)$. However, note from (6) that the logit differential equations imply that $f_1'(e_1) > f_2'(e_2)$, which yields the desired contradiction. Next, consider the case of equal prize values: $v_1 = v_2 = v$, for which we must show that the bid densities for players 1 and 2 are identical. Consider a particular effort level e. Since $F_2(e) >$

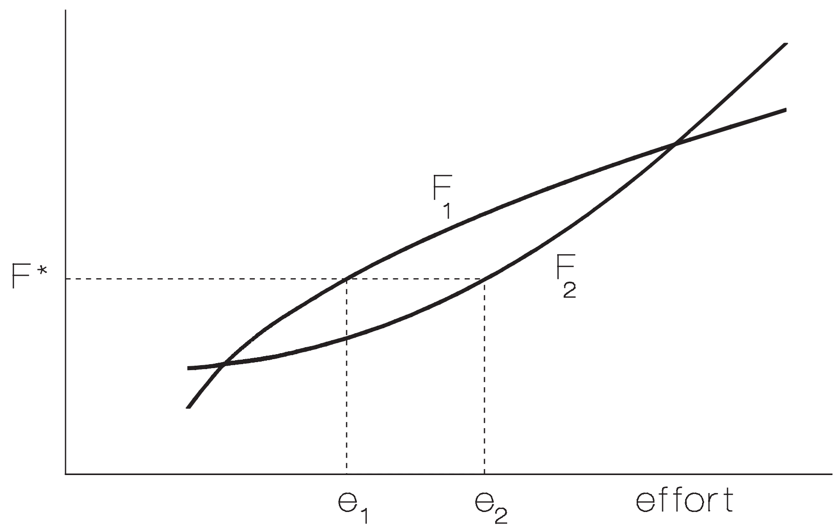

Figure 1. An Asymmetric Configuration

$F_1(e)$ for all $v_1 > v_2$, and $F_2(e) < F_1(e)$ for all $v_1 < v_2$, it follows from a continuity argument that the distributions are equal at e when $v_1 = v_2$. Obviously, this argument holds for all values of e. Q.E.D.

One interesting application of the logit equilibrium is that it provides a selection mechanism among different Nash equilibria. From Proposition 2 we know that the high-value player exerts more effort in a logit equilibrium, and in the limit that μ goes to zero, the logit equilibrium limits to the pure-strategy Nash equilibrium that has this property. In other words, the intuitive Nash equilibrium, in which the low-value player drops out, is selected. When players' values are equal and the maximum possible effort, $\bar{e}$, is infinite, the limit of the symmetric logit equilibrium is the symmetric mixed-strategy Nash equilibrium.[5] Next we show that the symmetric equilibrium is unique.

Proposition 3. The logit equilibrium is unique when values are identical.

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[5] Interestingly, the logit equilibrium does not limit to any Nash equilibrium when $\bar{e}$ is finite.

Proof. Given the symmetry result of Proposition 2, it suffices to show that there is at most one symmetric equilibrium. Suppose in contradiction that there are two symmetric equilibria, distinguished by "I" and "II" subscripts. Dropping the player-specific subscripts from (6) and using the derivative of the payoff in (2) yields the following differential equations for the two candidate solutions:

$$\begin{aligned} \mu f_I^{\prime} &= f_I\,(v f_I - c\,(1 - F_I)), \\ \mu f_{II}^{\prime} &= f_{II}\,(v f_{II} - c\,(1 - F_{II}))\,. \end{aligned} \tag{7}$$

Without loss of generality, we can assume $F_{\mathrm{II}}(x)$ is lower on some interval, as shown in Figure 2. Any region of divergence between the distribution functions will have a maximum *vertical* difference, as indicated by the vertical dashed line at e^*. The first and second order conditions for the distance to be maximized at height e^* are that the slopes of the distribution functions be identical, i.e. $f_{\mathrm{I}}'(e^*) = f_{\mathrm{II}}'(e^*)$, and that $f_{\mathrm{II}}'(e^*) \geq f_{\mathrm{I}}'(e^*)$. However, since $F_{\mathrm{II}}(e^*) < F_{\mathrm{I}}(e^*)$, the logit differential equations in (7) imply that $f_{\mathrm{II}}'(e^*) < f_{\mathrm{I}}'(e^*)$, which yields the desired contradiction. Q.E.D.

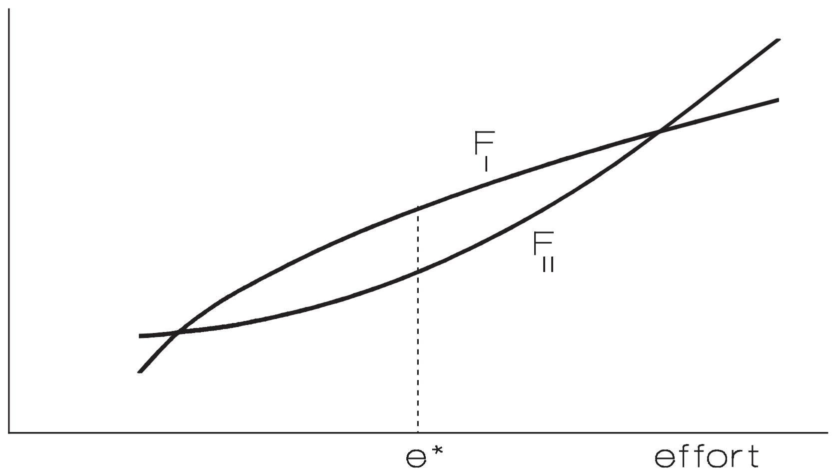

Figure 2. A Configuration that Yields a Contradiction when $F_I > F_{II}$.

The uniqueness result is quite surprising in light of the non-uniqueness of the Nash equilibrium. Uniqueness is a useful property because it ties down a particular prediction. Finally, we consider the effects of changes in the exogenous parameters on the equilibrium bid distributions. As would be expected, efforts are stochastically decreasing in the cost parameter, c, and stochastically increasing in the common prize value v.

Proposition 4. In the logit equilibrium for a symmetric normal-form war of attrition, efforts are raised (in the sense of first-degree stochastic dominance) by a decrease in the cost parameter, c, or an increase in the common value, v.

Proof. Let v_a be the common prize value for both players in one case, and let v_b be the common prize value in another case, where $v_a > v_b$. Similarly, let F_a and F_b denote the corresponding distributions. As before, the structure of the proof is to show that assume that F_a is greater than F_b on some interval. Therefore, there is a point of maximal horizontal distance, where $F_a(e_a) = F_b(e_b) = F^*$, $f_a(e_a) = f_b(e_b)$, and by the second-order condition $f_b'(e_b) \geq f_a'(e_a)$. It follows from the equations in (6) that the logit equilibrium densities satisfy

$$\mu f_a{}' \;=\; f_a\,(v_a f_a \;-\; c\,(1 - F_a)),$$
$$\mu f_b{}' \;=\; f_b\,(v_b f_b \;-\; c\,(1 - F_b))\,. \qquad (8)$$

Since $F_a(e_a) = F_b(e_b)$ and $f_a(e_a) = f_b(e_b)$, the right sides of the two equations in (8) are identical except for the prize values. Therefore, $f_a'(e_a) > f_b'(e_b)$ when $v_a > v_b$, which contradicts the second-order condition for the horizontal distance to be maximized. The proof that an increase in the common effort cost, *c*, decreases efforts, is similar. Q.E.D.

These comparative static results are consistent with the comparative static properties of the symmetric mixed-strategy equilibrium. However, our previous results for the mixed-strategy equilibrium suggests that the latter in this case is correct for the wrong reasons: an increase in one player's value alone has counter-intuitive effects by affecting only the other player's equilibrium density, but when both players' values rise the outcome is intuitive. In contrast, the logit equilibrium has the property that a change in each value alone has a direct effect on both players' equilibrium densities.

V. Conclusion

We have argued that the Nash equilibria for the second-price all-pay auction (or the normal-form war of attrition) have several unappealing properties. First, equilibria are typically not unique. This poses the problem that the equilibrium concept lacks predictive power (unless some suitable selection criterion is appended). Second, another interesting property of the Nash equilibrium is that the only equilibrium under asymmetric prize values and a bound on the maximum effort has one player choosing zero effort with probability one, although a symmetric mixed-strategy equilibrium exists when values are equal, and non-degenerate mixed-strategy equilibria exist for asymmetric values when there is no upper bound on the effort choice. In this sense an equilibrium type disappears when we move away from equal values and infinite effort bounds. Third, the mixed-strategy equilibrium for asymmetric prize values (and infinite maximum possible effort) has counter-intuitive properties: the player with the higher prize value exerts stochastically less effort, and an increase in one player's value causes no change in that player's effort choice, but causes the other player to exert more effort. In contrast, at any logit equilibrium the player with the higher prize value exerts more effort (Proposition 2) and the logit equilibrium is unique at least for identical values (Proposition 3).

The second-price all-pay auction has much in common with the entry game studied by Capra (1998a,b), and her experimental results shed some light on the war of attrition. In the entry game, two firms have to decide whether or not to enter a market. A monopoly is preferred by either to staying out, while if both firms enter they earn negative profits so that staying out is better. As Capra (1998a) notes, there are two pure-strategy equilibria (one player enters and the other stays out), as with the war of attrition. There is also a mixed-strategy equilibrium that has counter-intuitive comparative static properties: increasing one firm's entry cost leaves that firm's entry probability unchanged while decreasing the rival's entry probability, and, moreover, the firm with the higher entry cost is *more* likely to enter.

Capra (1998b) ran a series of one-shot experiments on the entry game with various asymmetric entry costs. She showed that the results of changing values are intuitive (and therefore opposite to those of the mixed-strategy Nash equilibrium). Indeed, her data "clearly show that a firm's entry probability is inversely related to its own entry costs and positive related to the rival's entry cost" (Capra, 1998b, p.16). Given the similarity between the entry game and the war of attrition, we should expect similar results from experiments on the latter, although these have not yet been done. One difference between the two games is that the entry game involves a simple binary choice while the war of attrition considers choice of an effort level from a continuum of possible choices.

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