Oligopoly Price Competition with Incomplete Information: Convergence to Mixed–Strategy Equilibria

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Abstract

This paper considers a market in which firms have incomplete information about a parameter of other sellers' payoff functions, e.g., a cost parameter. The dispersion of privately observed payoff parameters essentially enables firms to randomize their pricing. As the population distribution of payoff parameters collapses to a point, the distribution of pure-strategy prices converges to the mixed-equilibrium distribution for the limiting complete-information game. Both symmetric and asymmetric structures are considered.

A small discount from a common price can result in a large increase in the sales of a relatively homogeneous product, and the effect of this type of discontinuity is to make firms secretive about their price intentions. This secrecy would serve no purpose in a pure-strategy equilibrium (e.g., with significant product differentiation), but it is vital in either a mixed-strategy equilibrium or in a pure-strategy (Bayesian) equilibrium of a game with incomplete information.

In this paper, a standard incomplete-information model of a first-price auction is developed into an oligopoly model with price-setting firms. The incomplete information pertains to private, firm-specific random factors that alter firms' payoff functions (e.g., variations in costs or returns on alternative uses of excess capacity). Each firm's equilibrium strategy determines its price as a deterministic function of its own privately-observed payoff parameter.

The privately observed variations in payoff-function parameters effectively enable firms to adopt a deterministic pricing strategy that appears random to their competitors. As the parameter variation in the incomplete-information model vanishes, it is shown that the price

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distribution (determined by the pure-strategy equilibrium pricing rules) converges to the distribution for the mixed-strategy equilibrium of the limiting complete-information game. In this sense, mixed equilibria, which are often easier to calculate, can provide good approximations of Bayesian equilibria.

I. Introduction

In the absence of capacity constraints, firms selling a homogeneous product have an incentive to reduce price unilaterally at any price above marginal cost, and the unique Nash equilibrium is the competitive price. But capacity constraints can provide firms with unilateral market power (i.e., the ability to raise price profitably above a common competitive level).[1] In this case, the discontinuity caused by product homogeneity generally precludes the existence of a pure-strategy equilibrium price, and equilibria in mixed strategies have been analyzed extensively.[2]

Most of the interest in randomization arises from contexts in which pure strategies do not exist. In addition, some economists find randomized price to be plausible behavior because it generates price dispersion of the type commonly observed in many markets.[3] However, there is little evidence that individual sellers actually randomize. Jamie Kruse, Stephen Rassenti, Stanley Reynolds, and Vernon Smith (1990) did observe price distributions in laboratory experiments that approximate mixed equilibria, although pricing behavior was less random at the individual level. Colin Camerer and Keith Weigelt (1988) reported behavior that was roughly consistent with equilibrium mixing in a sequential reputation game, but when some subjects were debriefed, they forcefully denied randomizing. Robert Wilson (1969) first suggested a reason for this type of inconsistency between individual and aggregate behavior: privately observed random variations in individual firms' payoff functions effectively serve as a randomizing device in a

[1] Holt (1989) discusses this and other definitions of market power.

[2] See, for example, Kreps and Scheinkman (1983) for a characterization of mixed-equilibria in the presence of capacity constraints. Osborne and Pitchik (1986) contains a more general analysis and additional references.

[3] For example, Varian (1980) modeled the discounts offered during "sales" as randomized pricing. The Butters (1977) model of advertising can also be interpreted as an analysis of price dispersion produced by randomization.

game with incomplete information.[4] A large literature on auction games with incomplete information subsequently developed. These models have provided an important platform for the study of alternative auction institutions, although they typically lack the richness of demand and cost structures needed for many industrial-organization applications.[5]

The use of mixed equilibria has been motivated in other contexts as the limit of a sequence of Nash (Bayesian) equilibria for games with incomplete information as the individual payoff parameter variation vanishes. John Harsanyi (1973) first proved this type of convergence for the case of finite *N*-person games with additive payoff disturbances. Milgrom and Weber (1985) and Balder (1988) significantly generalized Harsanyi's convergence results to games with compact decision spaces. The relevant theorems in these papers are based on an assumption that the payoff functions are continuous in players' actions; this rules out the discontinuities that result from price competition with a homogeneous product.[6]

The original insights of Harsanyi and Wilson are developed in this paper in the context of a market price-setting game. Section II contains a parametric example. A more general model of price competition is presented in section III, and section IV contains a review of randomization for the case complete information. Privately observed payoff-parameter differences are introduced in section V, and the pure-strategy equilibrium for the game with incomplete information is characterized. Section VI contains a comparison of the equilibria in the preceding sections. Cost asymmetries are considered in section VII, and the final section contains a summary discussion.

II. A Simple Example

Suppose that there are two firms, each of which has the capacity to produce up to 2 units of output at a constant cost, which is normalized to zero. There is excess supply in the sense that

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[4] This was the motivation given by Wilson (1969) for the analysis of bidding for an oil lease as a game with incomplete information.

[5] Some representative papers are Wilson (1969), Holt (1980), Harris and Raviv (1981), Riley and Samuelson (1981), Cox, Roberson, and Smith (1982), Milgrom and Weber (1983), and McAfee and McMillan (1986). See Milgrom (1989) and Riley (1989) for introductory discussions of auctions and bidding.

[6] The "war-of-attrition" example in Milgrom and Weber (1985) does permit a discontinuity, however.

demand is inelastic at 3 units for any price below a reservation price r. Firms choose price simultaneously, so a firm with price p sells both units and earns $2p$ if it has the lowest price, and it sells only 1 unit and earns p otherwise. Demand is shared at any common price, so each firm has the usual Bertrand incentive to cut price unilaterally at any common price above the competitive level. But at a price of $r/2$, a firm would rather raise price all the way to r and sell 1 unit instead of selling 2 units at a price below $r/2$. In this sense, the range $[r/2, r]$ defines an "Edgeworth cycle", and no pure-strategy equilibrium exists. Suppose that a firm's competitor chooses price randomly using a continuous cumulative price distribution $G(\cdot)$. Then $G(p)$ is also the probability that the firm's own price p will be the higher price, and the resulting expected profit is: $G(p)p + [1-G(p)]2p$. A risk neutral seller would only be willing to randomize if this expected profit is constant over the range of prices being considered. The seller can obtain a security earnings of r from selling one unit at the reservation price, so the constant-expected-payoff condition is that $r = G(p)p + [1-G(p)]2p$, or equivalently:

(1) $$G(p) = \frac{2p - r}{p}.$$

Notice that the distribution function is increasing over the range of the Edgeworth cycle, with $G(r/2) = 0$ and $G(r) = 1$.[7]

Since the example is symmetric, when either firm mixes with distribution $G(p)$, the other firm's expected profits are constant on the range of the Edgeworth cycle. But when either firm uses the $G(p)$ function in (1), the other is indifferent between *all* probability distributions defined on $[r/2, r]$, so why would the other firm actually decide to use the same distribution function as its competitor?

In order to address this question, let firms observe a random variable that determines their own marginal costs, but each only knows the distribution of the other's cost parameter. By modifying the model in this way, each firm will have a unique, optimal price. In particular, each firm's cost is $c_i\delta$, where the c_i are realizations of a uniform random variable on $[-1, 1]$, and δ

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[7] Recall that cost was normalized to zero, so the distribution for a positive cost, c, could be obtained by adding c to both p and r in (1).

is the maximum deviation from the expected value of cost, which is normalized to zero. The δ is a scale parameter that will be reduced to zero in the limit.

In the equilibrium to be derived, each seller will choose a price that is determined as an increasing function of the cost parameter, c_i. Suppose that firm j is using a differentiable, strictly increasing function, $p_j = P(c_j)$, with $P(1) = r$.[8] Then firm i would have the highest price if $P(c_j) < p_i$, or equivalently, if $c_j < C(p_i)$, where $C(\cdot)$ denotes the inverse of the $P(\cdot)$ function. The uniform distribution of the cost parameters on $[-1, 1]$ implies that the probability of $c_j < C(p_i)$ is just $.5[C(p_i) + 1]$. Recall that the latter expression is the probability that firm i will have the highest price and sell only 1 unit, so the expected payoff for seller i can be expressed:

(2) $$.5[C(p_i) + 1][p_i - c_i\delta] + .5[1 - C(p_i)][2p_i - 2c_i\delta].$$

The first-order condition for expected payoff maximization can be expressed:[9]

(3) $$3 - C(p_i) - C'(p_i)[p_i - c_i\delta] = 0.$$

In a symmetric equilibrium, each seller will use the same strategy function, so $p_i = P(c_i)$, $C(p_i) = c_i$, and $C'(p_i) = 1/P'(c_i)$, for $i = 1, 2$. Making these substitutions into (3), one obtains a differential equation in the equilibrium pricing function:

(4) $$[3 - c]P'(c) - P(c) = -c\delta.$$

The left side of (4) is the derivative of $[3 - c]P(c)$ with respect to c. By integrating both sides of (4) from c_i to 1 and using the boundary condition that $P(1) = r$, one obtains:

(5) $$P(c_i) = \frac{2r + .5\delta(1 - c_i^2)}{3 - c_i}.$$

For any value of δ, the equilibrium strategy in (5) spans the entire range of the Edgeworth cycle (with no cost variation), i.e. $P(-1) = r/2$ and $P(1) = r$. As δ decreases and the cost variation diminishes, the pricing strategy is adjusted to maintain price unpredictability.

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[8] This boundary condition on the price strategy function is derived below.

[9] Sufficiency is discussed in section V and the appendix.

The next step is to compare the prices determined by (5) with the distribution of prices in (1) determined by randomization when costs are identical. In particular, consider the limiting distribution of prices determined by (5) as δ goes to 0. The convergence is uniform, and in the limit, $P(c_i) = 2r/(3 - c_i)$. It follows that the probability of $P(c_i) < p$ in the limit is the probability of $2r/(3 - c_i) < p$, or equivalently, the probability that $c_i < (3p - 2r)/p$. Since c_i is uniform on $[-1, 1]$, this probability is: $.5\left[(3p - 2r)/p + 1\right] = (2p - r)/p$, which is the mixed distribution derived above in (1). This example illustrates the sense in which small variations in the payoff parameters yield behavior that approximates randomization. Moreover, firms' prices are optimal given the privately observed cost parameters, which avoids the indifference over prices that many economists find troubling about a mixed-strategy equilibrium.

III. A Symmetric Model

The situation to be considered initially is a symmetric-cost, homogeneous-product duopoly, with risk-neutral firms that choose price simultaneously. A firm with price p and payoff parameter c will earn a profit denoted by $L(p,c)$ when it has the lower price, and a "residual" profit denoted by $H(p,c)$ when it has the higher price. In the event of a tie, the two firms will earn the average of the two. In the statements that follow, p and c subscripts denote partial derivatives. The payoff parameters are confined to a bounded interval $(\underline{c}, \bar{c})$, and Y denotes the set of (p, c) for which both payoff functions are strictly positive: $Y = \{(p,c);\ c \in (\underline{c}, \bar{c}),\ H(p,c) > 0,\ L(p,c) > 0\}$, which is assumed to be nonempty.

The specification of the H and L functions incorporates the ability of firms to restrict sales of units for which price is below marginal cost, and therefore, a firm can earn at least as much if its price turns out to be low as would be the case if its price were high. The first assumption makes this inequality strict.

Assumption 1. Both H and L are twice continuously differentiable, and $H(p,c) < L(p,c)$ for $(p,c) \in Y$.

Since the payoff in the event of a price tie is $(H + L)/2$, assumption 1 implies that the payoff at a common price p is less than L at that price, so a sufficiently small price cut will profitable. Assumption 2 implies that there is a price $\underline{p}(c)$, below which the best response is not to undercut the other's price, but rather, to raise price to the level, $p^*(c)$, that maximizes the residual profit function $H(p,c)$. This assumption generates an Edgeworth cycle on interval $[\underline{p}(c), p^*(c)]$.

Assumption 2. For (p,c) ε Y, the function H is concave in p and is uniquely maximized at $p^*(c)$. If $p < p^*(c)$, then $L_p > 0$. Moreover, there is a unique price $\underline{p}(c)$ such that $(\underline{p}(c), c)$ ε Y and

(6) $$H(p^*(c),\ c) = L(\underline{p}(c),\ c).$$

Next consider what assumptions are needed to ensure separation, i.e. that firms with higher values of c will prefer to set higher equilibrium prices in the Bayesian game. First, a positive sign for H_{pc} and L_{pc} makes price increases more attractive for firms with higher values of c, conditional on having the high or low price respectively. Second, a price increase makes the H function more likely, and this effect of a price increase is more attractive for firms with high c values if $H_c > L_c$. These conditions are summarized:

Assumption 3. For (p, c) ε Y, $H_{pc} \geq 0$, $L_{pc} \geq 0$, $L_c < H_c$, and H_c and L_c are bounded.

For example, if the c parameter represents a cost, then both H_c and L_c should be nonpositive, and the resulting condition, $L_c < H_c < 0$, implies that a cost increase is worse when the firm's price is low and quantity demanded is high.[10]

Consider a parametric example in which c represents a constant marginal cost up to a common capacity X. A quantity, Q, is demanded at any price below r, where $r > c$ and $X <$

[10] An alternative interpretation of the model could be that c represents the profit from the alternative use of extra capacity that results from having a high price (Holt, 1979a, 1979b). In this case it is natural to assume that $0 < L_c < H_c$, which is also consistent with assumption 3.

$Q < 2X$. The profit functions are: $L = (p - c)X$ and $H = (p - c)(Q - X)$. Therefore, $p^*(c) = r$, and (6) can be used to show that the lower limit of the Edgeworth cycle is a weighted average of c and r. If $X = Q$, then assumption 2 is violated, and the example becomes a standard model of bidding for a contract in which the purchaser has a reservation value of r and the high bidder earns nothing.

Now suppose that the example is modified so that market demand, $D(p)$, is downward sloping, industry total revenue, $pD(p)$, is concave, and there excess capacity at the lowest possible price: $2X > D(c)$. The firm with the lower price sells the units with the highest reservation values for buyers (a surplus-maximizing rationing rule). If $(p - c)D(p)$ is maximized at a price below D^{-1}(X), then it is never optimal for either firm to charge a price for which $D(p) < X$. Consequently, the low-price firm always sells its capacity and earns $L(p,c) = (p - c)X$, and the high-price firm sells the residual demand and earns $H(p,c) = (p - c)[D(p) - X]$. Here, $H_{pc} = -D'(p) > 0$ and $L_{pc} = 0$, which is consistent with the first part of assumption 3. The other condition, that $L_c < H_c$, implies that $2X > D(p)$, which is a consequence of the excess capacity assumption made above.

IV. Mixed-Strategy Equilibrium in a Symmetric Model[11]

If firms have a common payoff parameter, c then assumptions 1 and 2 imply that there is no pure-strategy equilibrium. With randomization, let $G(p)$ denote the probability that a firm's choice p will be the highest price, i.e. the probability that H is the relevant payoff function. Each firm must be indifferent between prices over which it is randomizing, so expected profits must equal a constant, V, for prices on the support, $[\underline{p}, \bar{p}]$.[12]

(7) $$V = G(p) \cdot H(p, c) + [1 - G(p)] \cdot L(p, c) \qquad for\ p\ \varepsilon\ [\underline{p}, \overline{p}].$$

For the duopoly case, $G(p_i)$ for firm i equals the probability that $p_j < p_i$, so $G(p)$ is the

[11] This section contains a standard analysis of randomized pricing that follows very closely the approach taken in Varian (1980), for example. Many of the arguments used here are standard, and the reader may wish to skip to the summary paragraph.

[12] Because the equilibrium to be considered is symmetric, there can be no probability mass points. Otherwise, each seller would prefer to reduce price slightly at a common mass point, which contradicts the constancy of expected profit on the support. Hence, the possibility of ties is not considered.

distribution function of the equilibrium price distribution of seller *j*, and vice versa. Therefore, equation (7) determines the symmetric price distribution, once *V* is found from an analysis of boundary conditions, which is the next task.

When both firms are randomizing on $[\underline{p}, \bar{p}]$, a choice of the highest price $\bar{p}$ will result in the *H* payoff function with certainty, so $\bar{p}$ should be the price that maximizes $H(p,c)$. It is straightforward to prove this observation, and therefore:[13]

$$(8) \qquad \bar{p} \;=\; \underset{p}{argmax}\; H(p,c) \;=\; p^{*}(c) \qquad and \qquad V \;=\; H\big(p^{*}(c),\; c\big).$$

The substitution of $H(p^*(c),c)$ for *V* in (7) yields a linear equation that is solved for $G(p)$. For purposes of comparison with the results of later sections, it is useful to express the solution in terms of normalized *H* and *L* functions, $H(p,0)$ and $L(p,0)$, where price is measured as a deviation from *c*:[14]

$$(9) \qquad G(p) \;=\; \frac{L(p,0) \;-\; H(p^{*}(0),0)}{L(p,0) \;-\; H(p,0)}.$$

Assumptions 1 and 2 can be used to show that this $G(p)$ is a valid distribution function with a lower bound $\underline{p}(0)$ that is determined by (6).[15]

To summarize, a firm must be indifferent between the prices over which it randomizes; equating the expected profit to a constant and analyzing the properties of the upper and lower boundaries yields the equilibrium probability of being the high-price firm, denoted $G(p)$ in (9).

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[13] As indicated in the previous footnote, there are no mass points in the symmetric equilibrium price distribution, i.e. the limit as $p \to \bar{p}$ of $G(p)$ is 1. Hence, (7) implies that $V = H(\bar{p}, c)$. If $\bar{p} < p^*(c)$, then the firm would prefer to raise price from the upper limit to $p^*(c)$, since $H(p^*(c), c) > H(\bar{p}, c) = V$. On the other hand, if $\bar{p} > p^*(c)$, then the expected profit at $p^*(c)$ would be a weighted average of two profits that each exceed *V*, since $L(p^*(c), c) > H(p^*(c), c) > H(\bar{p},c) = V$ by assumptions 1 and 2. Therefore, $\bar{p} = p^*(c)$.

[14] There is no loss of generality in such a normalization, since the change is additive and will not alter the properties assumed in assumptions 1-3.

[15] Since there are no mass points, $G(p) = 0$ at the lower bound, and then (8) and (9) imply that the lower bound is determined by (6). By assumption 2, *L* is strictly increasing in *p* for $p < p^*$, and equation (6) implies that the numerator in (9) is positive. The denominator in (9) is positive by assumption. The right side of (9) is less than 1 for $p < p^*(0)$ since *H* is uniquely maximized at p^* by assumption 2. The derivative of $G(p)$ has the same sign as $L_p[V - H] + H_p[L - V]$, which is positive since $L > V > H$ and both L_p and H_p are positive on $(\underline{p}, p^*)$. Therefore, $G(p)$ in (9) is a valid distribution function for the common equilibrium price distribution.

With a duopoly, $G(p)$ is also the distribution function for each firm's price, and assumptions 1 and 2 ensure that this function will be increasing. In an N-firm oligopoly, $G(p)$ in (9) is still the probability that the H function is relevant, but the determination of the symmetric equilibrium price distribution depends on which firms in the price ranking have earnings determined by the H function. For example, suppose that the quantity demanded is high enough so that the N-1 firms with the lowest prices obtain earnings determined by L, and the highest-price firm obtains H.[16] If $G_i(p)$ denotes the common price distribution for each firm in a symmetric equilibrium, then $G(p) = G_i(p)^{N-1}$, since the H is relevant when all $N-1$ other firms have lower prices. Then the equilibrium distribution is obtained by raising the right side of (9) to the power $1/(N\text{-}1)$.

V. Bayesian Equilibria in Posted-Price Auctions with Incomplete Information

In traditional models of bidding for contracts, the buyer has an inelastic demand for a fixed quantity specified in the contract. The winning (low) bidder obtains the contract profit, and the losers earn nothing. This auction model can be generalized to allow for a wider range of market outcomes in which the low-price firm sells all units that it is willing and able to provide, and the high-price firm sells the residual demanded. This section presents such a generalization of what is known in the auction literature as a private-values, first-price auction. The private value is denoted $c_i\delta$ for i = 1,2, where δ is a positive scalar and the c_i parameters are assumed to be independent realizations of a continuous random variable. The density and distribution functions for this variable will be denoted by $f(c_i)$ and $F(c_i)$ respectively, where $f(c_i) > 0$ on a compact support $[\underline{c}, \bar{c}]$. Following the normalization from the previous section, the private values are measured as deviations from a common level that is normalized to 0, so δ parameterizes the degree of inter-firm differences. The $c_i\delta$ terms represent the effects of firm-specific breakdowns, supply disruptions, productivity changes, or rental opportunities for idle capacity. Each firm

[16] Holt (1989) discusses a symmetric, N-firm model in which demand is inelastic at Q and X is a common capacity that satisfies: $(N\text{-}1)X < Q < NX$. The profit functions are: $L(p, c) = (p - c)X$ and $H(p, c) = (p - c)[Q - (N\text{-}1)X]$. Therefore, there is excess capacity at the competitive price, c, but profits are zero at this price, and each firm has the incentive to raise price unilaterally from a common competitive price in order to sell $Q - (N\text{-}1)X$ units at a positive profit margin of $p - c$. In this case of "unilateral market power", it is the firms with the $N-1$ lowest prices that sell their capacities of X, and the highest-price firm sells the residual. In contrast, only the lowest-price firm obtains sales to the "informed" buyers in Varian's (1980) model; the other $N-1$ firms sell a residual to their shares of "uninformed" buyers. In other contexts, there may be more than two profit functions, e.g., the functions can depend on the price ranking.

knows its own payoff parameter, but it only knows the common distribution from which the other's parameter is drawn. Thus firms have symmetric beliefs about each other's payoffs.

Since beliefs and payoff functions are symmetric but for the firm-specific payoff parameter, it is natural to consider an equilibrium that is symmetric: $p_i = P(c_i)$, i = 1,2, where the dependence of the pricing function on δ is suppressed for notional simplicity. The equilibrium is calculated by analyzing the optimal response of firm *i* when firm *j* is using the pricing rule $P(c_j)$, which is *assumed* to be strictly increasing and continuously differentiable on $(\underline{c}, \bar{c})$.[17] The inverse of the pricing function, denoted by $C(p_i)$, will also be strictly increasing. Consequently, firm *i* has the highest price if $P(c_j) < p_i$, or equivalently, if $c_j < C(p_i)$. The probability that firm *i* has the highest price will be denoted by $F_H(p_i)$, and therefore

(10) $$F_H(p_i) = Pr\{p_j \le p_i\} = Pr\{C(p_j) \le C(p_i)\} = Pr\{c_j \le C(p_i)\} = F\big(C(p_i)\big).$$

The far right-hand part of equation (10) shows the relationship between the firm's price and the probability that the *H* function is relevant.[18] Then the expected profit for firm *i* is:

(11) $$F\big(C(p_i)\big)\cdot H(p_i, c_i\delta) + \Big[1 - F\big(C(p_i)\big)\Big]\cdot L(p_i, c_i\delta),$$

where the mean of the c_i distribution has been normalized to zero in the *H* and *L* functions, i.e. all prices and costs are measured as deviations from *c*, as in the example in section I.

By differentiating the expected profit expression with respect to p_i and imposing symmetry, one obtains a differential equation in the equilibrium pricing function, $P(c)$, as was done in the example in section II.[19] There is a family of solutions to the differential equation for $P(c)$, and the relevant one is determined by a boundary condition at $\bar{c}$. In the symmetric equilibrium, the price $P(\bar{c})$ will be the highest price with certainty, so the equilibrium expected profit for $c = \bar{c}$ in (11) is $H(P(\bar{c}), \bar{c}\delta)$. But $P(\bar{c})$ must equal to the price, $p^*(\bar{c}\delta)$, that maximizes

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[17] It will be shown that the assumed properties of the pricing rule are satisfied in equilibrium.

[18] In a symmetric *N*-firm model, if the firm with the highest price is the only firm with earnings determined by the *H* function, then $F(C(p_i))$ in the subsequent expressions should be raised to the power $N-1$. This would not change any of the essential results that follow.

[19] This procedure confirms that the pricing function is increasing if H_p, L_p, and $[L - H]$ are positive on the relevant range of cost values, as required by assumptions 1 and 2.

H(p, $\bar{c}\delta$); i.e. the price that yields the H payoff function with certainty must maximize that function.[20]

Recall that the expression for the equilibrium mixed distribution in (9) for the complete information game was found by equating expected profit to a constant. This observation suggests that it may be helpful to consider an expression for the equilibrium expected profit function, $V(c)$, in the incomplete-information game. This function is obtained by replacing p_i with $P(c)$ in (11). By the envelope theorem, the derivative of the value function can be obtained by partially differentiating (11) with respect to c_i:[21]

(12) $$V'(c) \;=\; \delta\left[F(c)\cdot H_c \;+\; [1 \;-\; F(c)]\cdot L_c\right].$$

where the i subscripts have been dropped. Since the H payoff function is relevant at the highest price $P(\bar{c}) = p^*(\bar{c}\delta)$, the boundary condition for the value function is: $V(\bar{c}) = H(p^*(\bar{c}\delta),\ \bar{c}\delta)$. The integration of both sides of (12) from c_i to $\bar{c}$ yields:

(13) $$V(c_i) \;=\; H\left(p^*(\bar{c}\delta), \bar{c}\delta\right) \;+\; \delta\, I(c_i), \quad where$$

(14) $$I(c_i) \;=\; -\int_{c_i}^{\bar{c}}\left[F(c)\cdot H_c \;+\; [1 \;-\; F(c)]\cdot L_c\right] dc.$$

For the case in which the c_i represent parameters that are positively related to costs, both H_c and L_c will be negative, and therefore, $I(c)$ will be positive and decreasing in c_i. Since $I(\bar{c})$ = 0 and $H(p^*(\bar{c}\delta),\ \bar{c}\delta)$ is the expected payoff with $c_i = \bar{c}$, it follows from (13) that $\delta I(c_i)$ measures the benefit, in expected profits, of having a cost parameter below $\bar{c}$.[22] Equation

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[20] A strict inequality in either direction yields a contradiction: If $P(\bar{c}) < p^*(\bar{c}\delta)$, a firm with cost $\bar{c}$ could raise price above $P(\bar{c})$ and increase the value of H, which is still the relevant profit function when the price is increased above $P(\bar{c})$. If $P(\bar{c}) > p^*(\bar{c}\delta)$, a firm with cost $\bar{c}$ would prefer to reduce price from $P(\bar{c})$ to $p^*(\bar{c}\delta)$, since the expected profit at this price is a weighted average of two payoffs that exceed profits at $P(\bar{c})$: $L(p^*(\bar{c}\delta), \bar{c}\delta) > H(p^*(\bar{c}\delta), \bar{c}\delta) > H(P(\bar{c}), \bar{c}\delta)$ by assumptions 1 and 2. Therefore, $P(\bar{c})$ must equal $p^*(\bar{c}\delta)$.

[21] Even though equilibrium price is a function of c_i, the price derivative of (11) is zero in equilibrium.

[22] For the example from section I, recall that the high-price firm sells 1 unit and the low-price firm sells 2 units, so $H(p, c\delta) = p - c\delta$ and $L(p, c\delta) = 2(p - c\delta)$. For this example, equation (14) yields $I(c) = (c^2 - 6c + 5)$, which is zero when $c = \bar{c} = 1$ and positive for lower values of c.

(13) was derived from an analysis of first-order conditions, for $P(c_i)$ to be a Nash equilibrium, no deviation can be profitable.

Proposition 1. Under assumptions 1-3, any deviation from the pricing rule implicit in (13) and (14) will decrease expected profit.

A quick look at (11) should convince the reader that it would be complicated to compute a second derivative of expected profit, since a closed-form solution for the inverse pricing rule is not available. The proof of the proposition, which also applies to the *N*-firm symmetric case with an appropriate reinterpretation of $F(c)$, is given in the Appendix. The proof is constructed by considering a deviation to be an equilibrium price for the "wrong" payoff parameter, c_d, i.e., a firm with c_i chooses a price $P(c_d)$ that is not equal to $P(c_i)$. Then (13) and (14) are used to express the expected profit from deviation as the sum of $V(c_i)$ and a residual that is due to using the wrong value of cost parameter. In this manner, the expected profit from any deviation, not necessarily small, can be expressed in a way that is comparable to the formula for $V(c_i)$ in (13). The gain from deviation is shown to be negative if $L_c < H_c$ and if the cross partials, L_{pc} and H_{pc}, are positive as in assumption 3. The importance of these restrictions on the signs of the derivatives of the payoff functions generally been ignored in the auction literature, where a single unit is typically auctioned ($H = 0$) with a linear payoff function, ($L_{pc} = 0$).[23]

VI. A Comparison of the Mixed and Bayesian Equilibria

When $\delta = 0$ in the previous section's model, one obtains the symmetric model that was first considered in section III. For this symmetric model, the mixed-strategy equilibrium price distribution is determined by $G(p)$ in (9). Since $F_H(p)$ denotes the analogous equilibrium probability that p is the high price in the previous section's game with incomplete information, one natural question is whether $F_H(p)$ converges to $G(p)$ as firm-specific differences diminish,

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[23] Holt (1980) and McAfee and McMillan (1986) did analyze the sufficient conditions for a global maximum. Holt's model is one in which $L = p - K$ and $H = c_i$, where K is the cost of performing some specified job, and c_i is the best alternative profit for a firm's resources if it loses in the bidding. The McAfee and McMillan model is one in which $H = 0$ and L is an exponential function of a contract profit that is linear in price, cost, and a cost shock.

i.e. as $\delta \to 0$.

Equation (10) established the relationship between $F_H(p)$ and the exogenous distribution of firm-specific parameters, $F(\cdot)$; this relationship is: $F_H(p) = F(C(p))$, or by inverting the arguments, $F_H(P(c)) = F(c)$. But $F(c)$ can be determined by equating the solution for $V(c)$ from (13) with the expected profit in (11), evaluated at $p_i = P(c)$ and $c_i = c$. In this manner, the solution for $F(c)$, and hence for $F_H(P(c))$, is:

$$(15) \qquad F_H(P(c)) \;=\; F(c) \;=\; \frac{L(P(c), c\delta) - H(p^*(\bar{c}\delta), \bar{c}\delta) - \delta I(c)}{L(P(c), c\delta) - H(P(c), c\delta)},$$

which has the same structure as the formula for $G(p)$ in (9). One can use the inverse price rule, $c = C(p)$, to express (15) in terms of p:

$$(16) \qquad F_H(p) \;=\; \frac{L(p, C(p)\delta) - H(p^*(\bar{c}\delta), \bar{c}\delta) - \delta I(C(p))}{L(p, C(p)\delta) - H(p, C(p)\delta)}.$$

Equation (16) shows the probability that the other firm's price will be lower than a price of p in a symmetric equilibrium. As $\delta \to 0$, we know that $\delta I(c) \to 0$, since the derivatives in the integrand of (14) were assumed to be bounded. Then it is apparent that the distribution on the right side of (16) converges to the mixed strategy distribution in (9). To summarize:

Proposition 2. Under assumptions 1-3, as $\delta \to 0$, the equilibrium distribution of prices in the Bayesian game converges to the mixed-strategy equilibrium distribution for the symmetric complete-information game.

The intuition behind this result is simple. As δ goes to 0, the expected payoff advantage of having a relatively low c parameter diminishes ($\delta I(c) \to 0$), and $V(c)$ approaches a constant that is independent of c. As the equilibrium expected payoffs converge to this constant, the equilibrium is approximated by the constant-expected-payoff condition (7) that was used to calculate the symmetric mixed equilibrium in section III.[24]

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[24] This argument can be generalized to apply to the N-firm symmetric model with two payoff functions by transforming G and F_H in the same manner: If all firms but the one with the highest price earn payoffs calculated with

VII. Cost Asymmetries

The payoff variations parameterized by the c_i represent privately observed, firm-specific factors. In many industrial organization applications, however, there are publicly observed asymmetries. In this section, both the privately and publicly observed differences will pertain to costs, although the same technique can be used to analyze other types of asymmetries.[25] The section is organized around a discussion of: 1) the mixed-strategy equilibrium with complete information about cost asymmetries, 2) the pure-strategy Bayesian equilibrium for the asymmetric game with incomplete information, and 3) the convergence of the Bayesian equilibrium price distributions to the asymmetric mixed-strategy distributions.

The average cost for firm *i* will be represented as a function of the sum: $a_i + c_i\delta$, where the a_i are deterministic with $a_1 < a_2$, and the c_i are independent realizations of a continuous random variable with mean zero. (As before, think of all prices and costs as being measured as deviations from the mean of the c_i distribution.) Each firm knows the common distribution $F(c)$ for this random variable. The values of a_1 and a_2 are common knowledge, but only firm *i* is able to observe its own c_i. The a_i affect the profit functions that result from having the high or low price: $H(p,\ a_i{+}c_i\delta)$ and $L(p,\ a_i{+}c_i\delta)$, which are assumed to satisfy conditions analogous to assumptions 1-3 for each firm *i*.

Asymmetric Mixed Strategies[26]

First consider the equilibrium for the complete-information game with no privately observed cost variation ($c_1 = c_2 = 0$). Since $a_1 < a_2$, there will be *i* subscripts on the price $p_i^*(0)$ that maximizes the *H* function with $c_i = 0$. There will also be an *i* subscript on the price $\underline{p}_i(0)$

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the *L* function, for example, then $F_H = [F]^{N-1}$ and $G = [G]^{N-1}$ as discussed in section IV. Other ways of modeling the *N*-firm case are discussed in footnote 16.

[25] Considerable modifications are required if the firm-specific payoff parameters represent variations in capacity, however. This is because the residual demand of the high-priced firm can depend on the capacity of the other firm, i.e. the *H* function can depend on the other firm's payoff parameter.

[26] The results in this subsection are well known. Deneckere and Kovneock (1989) consider mixed equilibria for a duopoly with both cost asymmetries and capacity constraints. Holt and Solis-Soberon (1992) analyze mixed-strategy equilibria with asymmetric costs and with more than 2 sellers, for the case of rectangular demand (i.e. demand that is perfectly inelastic below some reservation price).

such that if both firms charge this price, then firm *i* is indifferent between a small price cut and an increase to the level that maximizes the residual payoff. For notational simplicity, the *c* argument of these price functions will be omitted when it is zero, so $p_i^*(0) \equiv p_i^*$ and $\underline{p}_i(0) = \underline{p}_i$. Then the analogue to (6) is:

(6') $$L(\underline{p}_i,\ a_i) \ = \ H(p_i^*,\ a_i).$$

In this section, the a_i are costs, which implies (17a) below. Since *a* and *c* appear additively in the payoff functions, the $L_c < H_c$ condition in assumption 3 implies that $L_a < H_a$, which implies (17b). Similarly, the nonnegativity of H_{pc} and L_{pc} yields the nonnegativity of L_{pa} and H_{pa}, which implies (17c) and (17d) respectively.

(17a) $$L(p, a_1) \ > \ L(p, a_2) \quad and \quad H(p, a_1) \ > \ H(p, a_2),$$

(17b) $$L(p, a_1) - H(p, a_1) \ > \ L(p, a_2) - H(p, a_2),$$

(17c) $$L_p(p, a_1) \ \leq \ L_p(p, a_2),$$

(17d) $$H_p(p, a_1) \ \leq \ H_p(p, a_2).$$

A final matter, assumption 2, which rules out pure-strategy equilibria, must be strengthened to ensure overlapping Edgeworth cycles for the asymmetric case. Recall that the underlined price, $\underline{p}$, determines the lower bound of the Edgeworth cycle in a symmetric model, and the starred price, p^*, determines the upper bound. In this section's asymmetric model, we assume that the highest lower bound is below the lowest upper bound. The upper bounds can be ranked using inequality (17d), i.e. the residual-profit-maximizing price is lower for the low-cost firm 1: $p_1^* \leq p_2^*$. Therefore, the necessary assumption is:

Assumption 4. Max $\{\underline{p}_1, \underline{p}_2\} < p_1^*$, and $L_p > 0$ for firm *i* at prices below p_i^*.

At any common parameter, $c_1 = c_2 = 0$, assumption 4 precludes a pure-strategy equilibrium in which one firm chooses a lower price that maximizes its L function and the other chooses a higher price that maximizes its H function.[27]

Now consider the mixed-strategy equilibrium when $a_1 < a_2$ and $c_1 = c_2 = 0$. Since the profit functions are strictly increasing in price below p_i*, neither firm would want to price below the other's lowest price, and there will be a common lower bound, denoted $\underline{p}$, for the support. Firms must be indifferent over all prices in the support of the mixed distribution, and the security values can be determined at the lower bound: $V_i = L(\underline{p}, a_i)$. This, together with (17a), implies that the firm with the lowest cost will have the highest value: $V_1 > V_2$. If $G_{Hi}(p)$ denotes the probability that firm i has the highest price, then the firm is indifferent over prices for which the constant-expected-payoff analog of (7) holds, which yields:

[27] Consider the case in which the low-cost firm 1 chooses a price, p_{L1}, that maximizes $L(p, a_1)$, and firm 2 chooses the price p_2^* that maximizes $H(p, a_2)$. This will be a pure-strategy Nash equilibrium if deviations are not profitable and if $p_{L1} < p_2^*$. Assumption 4 requires that $L_p(p, a_1) > 0$ at prices below p_1^*, so $p_{L1} \geq p_1^*$, and in addition, $p_1^* > \underline{p}_2$ by assumption 4. Therefore $p_{L1} > \underline{p}_2$ and $L(p_{L1}, a_2) > L(\underline{p}_2, a_2) = H(p_2^*, a_2)$, where the final equality follows from the definition of $\underline{p}_i$ in (6'). These relationships imply that firm 2 could slightly undercut the firm 1's proposed equilibrium price, p_{L1}, and obtain a higher profit, $L(p_{L1}, a_2)$, than the level, $H(p_2^*, a_2)$, that firm 2 earns in the proposed equilibrium.

(18) $$G_{Hi}(p) = \frac{L(p,a_i) - V_i}{L(p,a_i) - H(p,a_i)} \quad \textit{where} \quad V_i = L(\underline{p}, a_i).$$

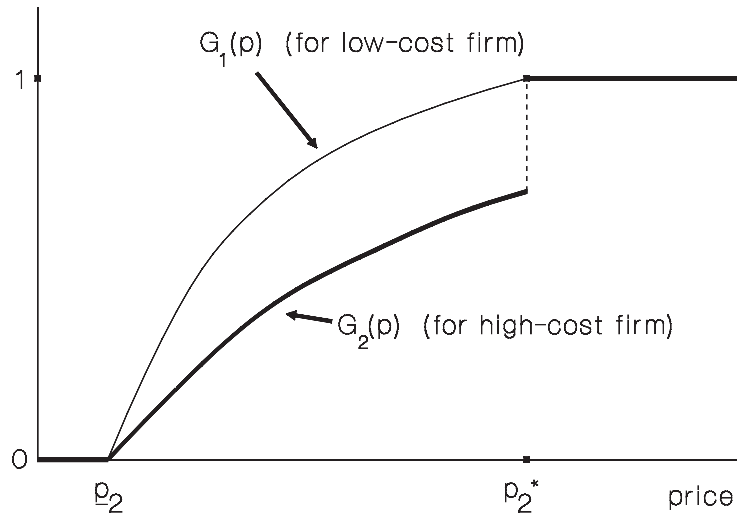

Figure 1. An Asymmetric Mixed-Strategy Equilibrium

To compare the price distributions of the two firms, note that the denominator of $G_{Hi}(p)$ in (18) is greater for the low-cost firm 1 by (17b). The numerator is expressed as $L(p,a_i) - L(\underline{p}, a_i)$, which is no greater for firm 1 by (17c). It follows that $G_{H1}(p) \leq G_{H2}(p)$, with strict inequality holding in the interior of the support. Recall that $G_{Hi}(p)$ is the probability that $p_i = p$ will be the higher price, so $G_{Hi}(p)$ is the distribution function of prices used by the *other* firm *j* in the mixed equilibrium: $G_{Hi}(p) = G_j(p)$. The result that $G_{H1}(p) \leq G_{H2}(p)$ corresponds to $G_1(p) \geq G_2(p)$, and therefore, the low-cost firm posts stochastically lower prices in the mixed equilibrium, as shown in figure 1.

Note that $G_1(p) > G_2(p)$ in the interior of the support, and as price increases, the $G_1(p)$ distribution function reaches a value of 1 first. The upper bound of the support of the mixed-strategy price distribution is the price at which $G_1(p) = 1$, as shown figure 1. Since the distribution function for firm 2 is lower at each price, firm 2 will have a mass of probability at the upper bound, $\bar{p}$, of the distribution. Then an argument analogous to footnote 13 can be used to show that $\bar{p} = p_2^*$.[28] In a mixed equilibrium, firm 2 must be indifferent between earning the residual payoff at the upper bound and the L payoff at the lower bound, $\underline{p}$, so

(19) $$H\left(p_2^*,\ a_2\right) \; = \; L(\,\underline{p},\ a_2).$$

It follows from (19) and (6') that $\underline{p} = \underline{p}_2$, and the security values are:

(20) $$V_2 \; = \; H(p_2^*,\ a_2) \; = \; L(\underline{p}_2,\ a_2) \qquad and \qquad V_1 \; = \; L(\,\underline{p}_2,\ a_1).$$

To summarize, the support of the mixed-strategy distributions is $[\underline{p}_2, p_2^*]$, and the distribution for the low-cost firm, $G_1(p)$, increases continuously from 0 to 1 on this interval. But the high cost firm 2 chooses stochastically higher prices with a mass of probability at the upper bound.[29]

Asymmetric Pure-strategy Price Functions

Next consider the game with incomplete information pertaining to the privately observed cost parameter; the H and L functions have c_i as arguments. The asymmetry of the a_i parameters suggests an asymmetric equilibrium with i subscripts on the price functions and their inverses: $p_i = P_i(c_i)$ and $c_i = C_i(p_i)$ for $i = 1,2$. These price functions will be strictly increasing on a range of c_i values to be determined. On this range, firm i has the higher price when $P_j(c_j) < p_i$, or equivalently, when $c_j < C_j(p_i)$, which occurs with probability $F\left(C_j(p_i)\right)$. Thus the expected payoff function for firm i is:

(21) $$F\left(C_j(p_i)\right) \cdot H\left(p_i, a_i + c_i\delta\right) \; + \; \left[1 \; - \; F\left(C_j(p_i)\right)\right] \cdot L\left(p_i, a_i + c_i\delta\,\right).$$

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[28] If $\bar{p}$ were below p_2^*, then firm 2 would prefer to raise price from the upper bound to the price p_2^* that maximizes the residual payoff function, which contradicts the Nash equilibrium condition. On the other hand, suppose that $\bar{p}$ were strictly greater than p_2^*. Assumption 1 and the fact that p_2^* maximizes the residual payoff function imply that $L(p_2^*, a_2) > H(p_2^*, a_2) > H(\bar{p}, a_2)$. Thus the expected payoff at p_2^* would be a weighted average of two payoffs that each exceed the certain payoff, $H(\bar{p}, a_2)$, for the mass of probability at the upper bound, which again violates the Nash condition. Hence $\bar{p} = p_2^*$.

[29] It is possible to have gaps in the support of the mixed distribution. To see this, note that the slope of $G_{H1}(p)$ agrees in sign with $L_p[V_1 - H] + H_p[L - V_1]$, and this slope may be negative since $H_p(p, a_1) < 0$ for $p > p_1^*$. A negative slope over this region would generate a gap in the support of the distributions; in what follows it is assumed that $H(p, a_1)$ is sufficiently flat at prices between p_1^* and p_2^* to avoid this irregularity. For example, with a demand that is perfectly inelastic at prices below r, $p_1^* = p_2^* = r$, and this problem does not arise.

The first-order condition is obtained by differentiating (21) with respect to p_i:

(22) $$F\big(C_j(p_i)\big)H_p \;+\; \big[1 - F\big(C_j(p_i)\big)\big]L_p \;-\; f\big(C_j(p_i)\big)C_j'(p_i)[L - H] \;=\; 0.$$

Since both payoff functions are increasing in price in the relevant range, neither firm would want to charge a price that is below the lowest possible price that the other would charge, i.e. the price for a cost of $\underline{c}$. Thus at $c = \underline{c}$, both firms choose the same price, which will be denoted by $\underline{p}$. But at $c > \underline{c}$, the low-cost firm would choose a lower price:

Proposition 3. Under assumptions 1-4, if firm 1 has a lower expected value of average cost below capacity ($a_1 < a_2$), then $P_1(c) < P_2(c)$.

Proof. Consider equation (22) evaluated at $p_i = \underline{p}$, $c_i = \underline{c}$, $C(p_i) = C_j(p_i) = C_j(\underline{p}) = \underline{c}$. This equation can be solved for the derivative of $C_j(\cdot)$:

(23) $$C_j'(\underline{p}) \;=\; \frac{F(\underline{c})\cdot H_p(\underline{p},\ a_i + \delta\underline{c}) \;+\; [1 - F(\underline{c})]\cdot L_p(\underline{p},\ a_i + \delta\underline{c})}{f(\underline{c})\big[L(\underline{p},\ a_i + \delta\underline{c}) - H(\underline{p},\ a_i + \delta\underline{c})\big]}.$$

It follows from the relationships in (17b-17d) that the H_p and L_p terms in the numerator of (23) are lower for firm 1, and that the $[L - H]$ difference in the denominator is greater for firm 1. Hence (23) implies that $C_2'(\underline{p}) < C_1'(\underline{p})$, or equivalently, $P_2'(\underline{c}) > P_1'(\underline{c})$. Since the two price functions begin at the same point $(\underline{c}, \underline{p})$, it follows that $P_2(c) > P_1(c)$ in a neighborhood of the lower limit, $\underline{c}$. These two price functions are continuous, and they cannot intersect, since the analog to (23) at the point of intersection (c,p) implies that $P_2'(c) > P_1'(c)$, and so $P_2(c)$ cuts $P_1(c)$ from below, which is a contradiction. ■

Since $P_2(c) > P_1(c)$ for low values of c, and since it was firm 2 that has the mass of probability at the highest price in the mixed-strategy equilibrium, it is natural to consider a configuration of equilibrium pricing functions as shown in figure 2, where $\bar{p} = P_1(\bar{c}) = P_2(\bar{c})$. Here firm 2 has a mass of probability at $\bar{p}$, i.e., $P_2(c) = \bar{p}$ whenever c is to the right of $\hat{c}$ in the figure. In this configuration, firm 2, but not firm 1, has the higher price with probability 1 when it chooses a price of $\bar{p}$, and therefore, the price $\bar{p}$ for firm 2 always results in the H payoff

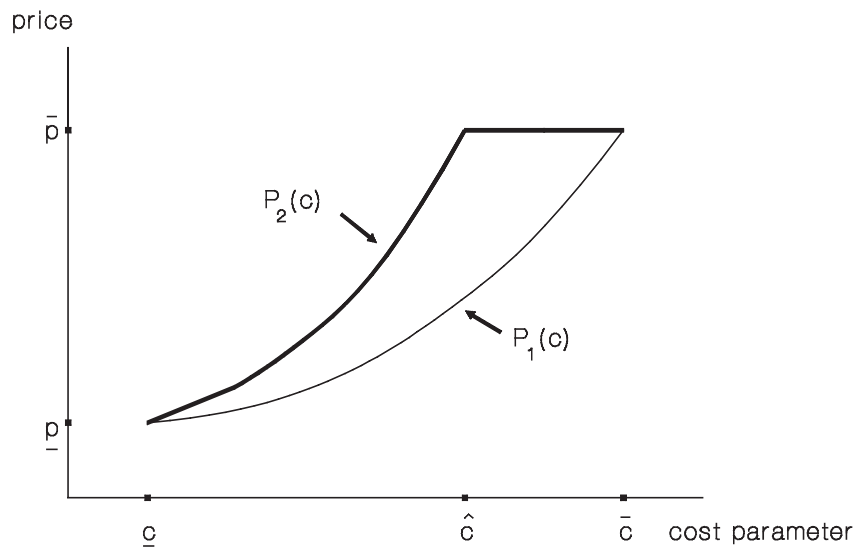

Figure 2. Asymmetric Pure-Strategy Pricing Functions

function and must maximize that function: $\bar{p} = p_2*(\bar{c}\delta)$.[30]

Now consider a comparison between the pure-strategy equilibrium represented in figure 2 and the mixed-strategy equilibrium with $c_i = c_i = 0$. The procedure will parallel the approach taken in the last section: it will be shown that the equilibrium expected payoffs converge to the same constants that appear in the constant-expected payoff conditions for the mixed-strategy equilibrium.

The first step is to use the envelope-theorem argument: Let $V_i(c_i)$ represent the equilibrium expected payoff for seller *i*. Then the derivative of $V_i(c_i)$ with respect to c_i is just the partial derivative of the expected payoff in (21):

(24)
$$V_i'(c_i) \;=\; \delta\Big[F\big(C_j(P_i(c_i))\big)\cdot H_c \;+\; \big[1 \;-\; F\big(C_j(P_i(c_i))\big)\big]\cdot L_c\Big].$$

Recall that neither firm will want to select a price that will be strictly below the other's lowest price, so there is a common lower bound, $P_i(\underline{c}) = \underline{p}$. At this lower bound, $V_i(\underline{c}) = L(\underline{p}, a_i{+}\delta\underline{c})$. The equilibrium expected profit for higher values of c_i can be obtained by adding the integral of

[30] Clearly, $\bar{p}$ must be greater than or equal to the price $p_2*(\bar{c}\delta)$ that maximizes $H(p, a_2 + \delta\bar{c})$. If $\bar{p}$ were strictly greater than $p_2*(\bar{c}\delta)$, then H_p would be negative at $\bar{p}$ by the strict concavity of H in assumption 3. Since $C_1(\bar{p}) = \bar{c}$ for the configuration in figure 2, $1 - F(C_1(\bar{p})) = 0$, and (24) would imply that $C_2'(\bar{p}) < 0$, which is a contradiction.

the $V'(c)$ derivative in (24) to this expression for $V_i(\underline{c})$:

(25) $$V_i(c_i) = L(\underline{p},\ a_i + \delta \underline{c}) + \int_{\underline{c}}^{c_i} V_i'(c)\,dc\,,$$

As the dispersion parameter, δ, goes to zero, V_i' in (24) converges to 0, and $V_i(c_i)$ in (25) converges to a constant $L(\underline{p}, a_i)$ for all values of c_i. The next step is to determine $\underline{p}$ so that $L(\underline{p}, a_i)$ can be used to calculate the limiting values of the expected profit function for each seller.

Consider the top of the price distribution, where firm 2 has a mass of probability. Because firm 2 has the highest price with probability 1 when $c_2 = \bar{c}$, it is the case that $V_2(\bar{c}) = H(p_2^*(\bar{c}\delta),\ a_2+\delta\bar{c})$. Therefore $V_2(\bar{c})$ converges to $H(p_2^*, a_2)$ as δ goes to zero.[31] It was shown in the previous paragraph that V_2 also converges to $L(\underline{p}, a_2)$, for all values of c_2. By equating these two expressions, one obtains an equation, $L(\underline{p}, a_2) = H(p_2^*, a_2)$, that corresponds to (6'), which implies that the limiting value of the lower bound is the price $\underline{p}_2$ determined by (6') for firm 2. Therefore the limiting values of the V_i are $L(\underline{p}_2, a_i)$, which are the same expected profit constants in (20) that were used to calculate the mixed equilibrium. Equating these limiting values of the V_i to the expected profit in (21) yields equations that produce the same distributions as the mixed distributions in (18) and (20) above. Moreover, the price support in the incomplete-information game converges to $[\underline{p}_2, p_2^*]$, which corresponds to the support of the mixed equilibrium in the game with complete information (cf. figure 1).

VIII. Conclusion

Perhaps the simplest model of market competition is one with price-setting firms and a homogeneous product. This competition is modeled as a game in which firms have incomplete information parameters of about others' payoff functions. The pure-strategy equilibrium is characterized by a price function that maps a firm's private payoff parameter into the price. In asymmetric models, each firm has a commonly known payoff parameter, a_i, in addition to the privately observed parameter, c_i, and the price functions differ. For example, if a firm's average

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[31] Recall the notational simplification used in (6'): $p_2^*(0) \equiv p_2^*$.

cost is $a_i + c_i\delta$, then the firm with the lower value of a_i selects a lower equilibrium price for each realization of c_i. As the distribution of the privately observed cost parameters collapses toward a point, the pricing function becomes more sensitive to cost variations in a way that magnifies the effects of payoff differences, i.e. firms essentially use small payoff variations as a mechanism for randomizing price. As the variation in the payoffs vanishes, the distribution of observed equilibrium prices converges to the distribution that would result from a mixed-strategy equilibrium in the limiting game with complete information. The intuition is that "small" variations in private payoff parameters can be used to spread prices strategically, but equilibrium expected payoffs are approximately constant for the small range of payoff parameters encountered. In the limit, this approximate constancy of expected payoffs in the pure-strategy equilibrium corresponds to the mixed-strategy equilibrium condition that sellers be indifferent over all prices in the range of randomization.

Appendix: Proof of Proposition 1

A firm with c_i that considers a deviation price $p_d \neq P(c_i)$ can be thought of as choosing a price $P(c_d)$ for which implicit cost c_d is not equal to the firm's true cost. The deviation price will be high with probability $F(c_d)$, so the expected profit from deviation, denoted $D(c_d)$, is:

(A1) $$D(c_d) \;=\; F(c_d)\cdot H\big(P(c_d), c_i\big) \;+\; \big[1 - F(c_d)\big]\cdot L\big(P(c_d), c_i\big),$$

where $\delta = 1$ with no loss of generality.

The first step is to relate the deviator's expected profit to $V(c_d)$, which is the right side of (A1) evaluated at $c_i = c_d$; this is done by adding and subtracting this expression for $V(c_d)$ to the right side of (A1) to get:

(A2) $$\begin{aligned} D(c_d) \;=\; & V(c_d) \;+\; F(c_d)\cdot\Big[H\big(P(c_d),\, c_i\big) - H\big(P(c_d),\, c_d\big)\Big] \\ & +\; [1 - F(c_d)]\cdot\Big[L\big(P(c_d),\, c_i\big) - L\big(P(c_d),\, c_d\big)\Big]. \end{aligned}$$

The terms in large brackets in (A2) can be expressed as integrals of partial derivatives, to yield:

(A3) $$D(c_d) \;=\; V(c_d) \;+\; \int_{c_d}^{c_i}\Big[F(c_d)\cdot H_c\big(P(c_d), c\big) + \big[1 - F(c_d)\big]\cdot L_c\big(P(c_d), c\big)\Big]\, dc.$$

The net profit from deviation is found by subtracting $V(c_i)$ from the right side of (A3). This net profit can be simplified by using (13) to eliminate the common term $H\big(p^*(\bar{c}), \bar{c}\big)$ from $V(c_i)$ and $V(c_d)$, and by expressing the remaining terms as an integral. For the case in which the deviation involves a price reduction, $c_i > c_d$, this integral is:

(A4) $$D(c_d) - V(c_i) \;=\; \int_{c_d}^{c_i}\big[\, A(c, c_d) - B(c)\,\big]\, dc, \quad \textit{where}$$

(A5) $$A(c, c_d) \;=\; \Big[F(c_d)\cdot H_c\big(P(c_d), c\big) + \big[1 - F(c_d)\big]\cdot L_c\big(P(c_d), c\big)\Big].$$

(A6) $$B(c) \;=\; \Big[F(c)\cdot H_c\big(P(c), c\big) + \big[1 - F(c)\big]\cdot L_c\big(P(c), c\big)\Big].$$

Since $A(c, c_d)$, evaluated at $c = c_d$, equals $B(c_d)$, and since $c > c_d$ over the interval of integration, the integrand will be negative if the derivative of $A(c, c_d)$ with respect to c_d is positive:

(A7)
$$\Big[F(c_d)\cdot H_{pc}\big(P(c_d), c\big) + \big[1 - F(c_d)\big]\cdot L_{pc}\big(P(c_d), c\big)\Big]P'(c_d)$$
$$+ f(c_d)\Big[H_c\big(P(c_d), c\big) - L_c\big(P(c_d), c\big)\Big] > 0,$$

for $c_i > c > c_d$. Since $H_c > L_c$ and both H_{pc} and L_{pc} are positive by assumption 3, the condition in (A7) is satisfied. Therefore, no price reduction from the price determined by $P(c_i)$ is attractive.

In the case of an upward price deviation, $c_d > c_i$, the limits of integration in (A4) are reversed, and the integrand becomes $B(c) - A(c, c_d)$. It is straightforward to show that the reversed relationship between c_d and c over the limits of integration causes both $B(c) - A(c, c_d)$ and the gain from deviation to be negative. ■

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