Rent Seeking with Bounded Rationality: An Analysis of the All-Pay Auction

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ABSTRACT

The winner-take-all nature of all-pay auctions makes the outcome sensitive to decision errors, which we introduce with a logit formulation. The equilibrium bid distribution is a fixed point: the belief distributions that determine expected payoffs equal the choice distributions determined by expected payoffs. We prove existence, uniqueness, and symmetry properties. In contrast to the Nash equilibrium, the comparative statics of the logit equilibrium are intuitive: rent dissipation increases with the number of players and the bid cost. Over-dissipation of rents is impossible under full rationality, but is observed in laboratory experiments. Our model predicts this property.

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I. Introduction

Many economic allocations are decided by competition for a prize on the basis of costly activities. For example, monopoly licenses may be awarded to the person (or group) that lobbies the hardest (Tullock, 1967), or tickets may be given to those who wait in line the longest (Holt and Sherman, 1982). In such contests, losers' efforts are costly and generally not compensated. These situations, which are especially common in non-market allocations, are of concern to economists precisely because competition involves the expenditure of real resources, or "rent-seeking" behavior. Krueger (1974) estimated that the annual welfare costs of rent seeking induced by price and quantity controls to be 7 percent of GNP in India, and somewhat higher in Turkey. Mohammad and Whalley (1984) reconsidered the cost of rent seeking in India and came up with much larger estimates, on the order of 30-45 percent of GNP. They conclude that "... these numbers put rent seeking in India into an entirely different category from more traditional policy issues such as trade liberalization, tax reform, and the like...." (Mohammad and Whalley, 1984, pp.387-388). In the U.S., Posner (1975) estimated the social cost of regulation to be up to 30 percent of sales in some industries (motor carriers, oil, and physicians' services).[1]

Following Tullock (1980), the literature on rent seeking is based on the assumption that the probability of obtaining the prize is an increasing function of one's own effort. The limiting case in which the prize is always awarded to the competitor who exerts the highest effort is called an "all-pay auction." The auction formulation applies when efforts are like monetary bids that can be ranked easily, as is the case with awarding tickets to those who wait the longest in line, or choosing a weapon system on the basis of easily measured performance criteria. The all-pay assumption is commonly used in the literature on lobbying (Hillman, 1988, and Hillman and Samet, 1987), since expenditures incurred in the competition for a government grant, license, or contract are usually not reimbursed. Other applications of the all-pay auction include R&D races, political contests, and promotion tournaments.

The prize goes to the highest bidder in an all-pay auction, so each bidder has an incentive to bid just above the highest of the others, as long as this allows a positive payoff. Therefore, there is typically no equilibrium in pure strategies. In symmetric all-pay auctions, the mixed-

[1] All of these estimates are based on the assumption that rents are fully dissipated. The analysis can also be criticized on other grounds, but the magnitude of these figures suggests the importance of reducing rent seeking.

strategy equilibria involve full dissipation of the rent, i.e., the sum of the expected bids equals the value of the prize (Baye, Kovenock, and de Vries, 1996). In particular, rents could never be over-dissipated with rational players who can always ensure a zero payoff by bidding zero. However, over-dissipation might occur when people are not perfectly rational. Davis and Reilly (1994) report a pervasive pattern of over-dissipation for all-pay auctions in laboratory experiments with financially motivated subjects.

This paper develops a theoretical model in which bidding behavior is subject to error. The introduction of errors is motivated by the observation that behavior in laboratory experiments can be "noisy" (e.g., Bull, Schotter, and Weigelt, 1987; Smith and Walker, 1993b, 1997) and can systematically deviate from Nash predictions.[2] To put this into perspective, recall that a Nash analysis has two components: perfectly rational decision making and consistency of beliefs and decisions. Here, we relax the assumption of perfect rationality, while keeping the consistency of beliefs and decisions.[3]

Our approach should be thought of as an equilibrium analysis with boundedly rational players.[4] Bid decisions are assumed to be determined by expected payoffs via a logit

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[2] Deviations from Nash predictions are summarized in chapters 2, 5, and 6 of Davis and Holt (1993).

[3] Another possibility is to relax the consistency of beliefs and decisions. This raises the related issue of learning and adjustment to equilibrium, as proposed by Sargent (1993, p.3). In Anderson, Goeree, and Holt (1997), we specify a stochastic evolutionary model, for which the steady state is a logit equilibrium. In particular, players are assumed to adjust their decisions in the direction of increasing payoffs, subject to some randomness. The variance of the noise determines the error parameter in the logit equilibrium. Alternatively, Chen, Friedman, and Thisse (1995) show convergence to a logit-type equilibrium in a model of naive learning (fictitious play) when players make decision errors that are determined by a probabilistic choice rule. Offerman, Schram, and Sonnemans (1996) and Brandts and Holt (1995) show that naive Bayesian learning, together with logit decision error, provides a good explanation of the patterns of adjustment in data from laboratory experiments with step-level public goods and signaling games. McKelvey and Palfrey (1996) provide a theoretical logit analysis of a step-level public goods game.

[4] Smith and Walker (1993b, 1997) model decision error as noise around a target decision level. This approach could be thought of as "implementation error," in that players know they (or their agents) will imprecisely implement desired actions. In the Smith and Walker model, players can decrease the variance of errors at a cost. The model predicts that scaling up payoffs should shift the average outcome toward that of rational play (with higher payoffs, players increase effort to reduce the error variance), and decrease the variance of outcomes. (Similar properties hold under our approach.) Smith and Walker (1993b) then show that these predictions are broadly consistent with results from a survey of 31 experimental studies. Smith and Walker (1993a) provide further evidence for these hypotheses (in the context of a first-price auction). One difference between their approach and ours is that errors in their model are assumed to be centered around the Nash equilibrium (with no error): the actual decision is equal to the Nash equilibrium decision plus a random error with zero mean and a variance that is determined by a costly effort. In contrast, the equilibrium distribution of decisions in our model is not necessarily centered around the Nash equilibrium, and systematic biases can occur even when the Nash equilibrium is not at a boundary of the set of feasible actions.

probabilistic choice rule, where decisions with higher expected payoffs are more likely to be chosen, although not with probability one. The sensitivity of decisions to payoff differences is determined by an error parameter that allows perfect rationality as a limiting case. The equilibrium is a fixed point in probability distributions: the bid distributions determine the expected payoffs for each bid, which in turn determine the probability distributions of actual bids. The model is closed by requiring that the belief distributions correspond to the decision distributions. This "Nash plus logit" approach has been termed a *logit equilibrium* by McKelvey and Palfrey (1996). The logit equilibrium is a stochastic generalization of the Nash equilibrium and a special case of the quantal response equilibrium proposed by McKelvey and Palfrey (1995).[5]

One especially appealing feature of the incorporation of errors into the equilibrium analysis is that comparative statics properties are, in some cases, more intuitive than for the standard Nash analysis (with no error). If two bidders' prize valuations are known and different, for example, then an increase in value for the player with the higher value will stochastically increase that player's bids. In contrast, this increase in the high value will not affect the equilibrium bid density of the high-value bidder in the mixed-strategy Nash equilibrium. In other games, the logit equilibrium also provides a plausible explanation of data patterns that are consistent with economic intuition but are not predicted by a Nash equilibrium.[6]

For the all-pay auction, Lopez (1995, 1996) identifies conditions under which the logit and Nash equilibria are identical, and there is exact dissipation of rents in both cases. This equivalence only holds with two bidders, identical prize values, and a maximum allowed bid that

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[5] Rosenthal (1988) pioneered the use of probabilistic choice in an equilibrium framework: he essentially used a linear probability model instead of a logit formulation. Our approach is closer to that of Lopez (1995), who considers Bertrand and auction games with continuous choice variables. Lopez uses both the logit rule and a ratio choice function that was first proposed by Luce (1959).

[6] In public goods games where free-riding is a dominant strategy, the level of voluntary contributions in laboratory experiments is increasing in the marginal value of the public good. The logit equilibrium model explains both this and other anomalous patterns in the data (Anderson, Goeree, and Holt, 1996a). Palfrey and Prisbrey (1996) use an ordered Probit analysis of the data from a public goods experiment, and they conclude that errors are significant. The logit model also explains data patterns in continuous coordination games, i.e., an increase in the cost of "effort" or in the number of players reduces observed effort levels (Anderson, Goeree, and Holt, 1996b). This pattern is consistent with economic intuition, but not with Nash predictions (since any common effort level is a Nash equilibrium). McKelvey and Palfrey (1995) evaluate data from several normal-form games, and reject the Nash equilibrium in favor of the logit equilibrium.

equals the common value. We show that over-dissipation is possible in the logit equilibrium when these assumptions are relaxed, e.g., when there are more than two players. In particular, the logit equilibrium model provides an explanation of over-dissipation of rents observed by Davis and Reilly (1994) in laboratory experiments. Moreover, the model allows us to analyze the tradeoff between the costs of rent seeking and the benefit from allowing the high-value bidder to compete more aggressively for the prize.

The model is described in the next section, and the equilibrium for the two-player case is analyzed in section III. Some key properties of the *n*-player all-pay auction (existence, uniqueness, symmetry, and comparative statics) are derived in section IV. On a first reading, one may wish to skip the proofs in section IV, and go on to the analysis of rent seeking and efficiency in section V. Section VI concludes.

II. The Model

In an *n*-player all-pay auction, player i bids b_i for a prize that is worth V_i dollars to player i. Bids are made simultaneously. The prize goes to the highest bidder, but each player incurs the cost of bidding, cb_i, where $c > 0$.[7] In the event of a tie for the highest bid, the prize is either split equally or randomly allocated to one of the high bidders. This model can be interpreted as a lobbying game in which c is the cost of lobbying effort that must be borne whether or not the effort is successful. In many contexts, a maximum allowable bid, B, is specified by the rules of the auction or is implied by resource constraints. For instance, subjects in a laboratory experiment may not be allowed to bid more than their initial cash endowments given out at the beginning of the experiment. In a Nash equilibrium (without errors), the cost associated with the maximum observed bid will never exceed the prize value, since higher bids are dominated by a bid of zero. Therefore, in a Nash analysis of a symmetric model, there would be no loss of generality in assuming that the maximal allowable bid is equal to V/c. However, in some

[7] In most theoretical work, as in laboratory experiments, c is equal to 1. We chose not to normalize, in order to consider the effects of independent changes in the bid cost, and of differences in individuals' bid costs. Moreover, normalizations are not innocuous in a logit analysis. A multiplicative change in payoffs could be used to normalize c to 1. This would not affect the Nash equilibrium (with no error) since any payoff difference, no matter how small, will determine which decision is taken. However, doubling payoffs will double all payoff differences, which reduces the impact of errors in the logit model specified below (by halving the error parameter μ).

laboratory experiments with $c = 1$, bids above value have been observed (Davis and Reilly, 1994). To permit this kind of error, we allow the cost of the maximum bid to exceed the prize value, i.e., $cB \geq V$, although our comparative static and characterization results apply also to the case $cB < V$. In particular, we show that over-dissipation can occur even when the cost of the maximum allowable bid is less than the prize value.[8]

The first step in the analysis is to establish the connection between a player's expected payoffs and the others' bid distributions. Let $F_j(b)$ denote the cumulative bid distribution for player j, so the probability of winning with a bid of b is the probability that all other bids are below b, i.e., the product of the others' distribution functions evaluated at b. Thus the expected payoff for player i, for a bid b, is:[9]

$$\pi_i^e(b) = V_i \prod_{j \neq i} F_j(b) - cb, \qquad i = 1, \ldots, n. \tag{1}$$

The second step is to introduce decision error, by specifying the bid density as an increasing function of expected payoff, but without having all of the probability located at the bid that maximizes the expected payoff. The logit form is one particularly useful parametric model of such probabilistic choice; it specifies decision probabilities to be proportional to an exponential function of expected payoffs.[10] In the continuous version of the logit model, the bid density is exponential in expected payoffs: $f_i(b) = k_i \exp(\pi_i^e(b)/\mu)$, where μ is an error parameter and k_i is a constant that ensures that the density integrates to one. Since expected payoff in (1) is zero at $b = 0$, it follows that $k_i = f_i(0)$. The expected payoff in (1) is finite for all possible bids, so the logit density is finite for all non-zero values of μ. Therefore, the resulting distribution functions are continuous, and the probability of ties is zero in a logit

8 See the discussion following Proposition 6. Note, however, that over-dissipation is precluded by assumption when the maximum bid, B, is less than V/cn, since then the maximum total effort is $ncB < V$.

9 The right side of (1) would have to be modified if ties occurred with positive probability. We show below that the logit equilibrium density is continuous, so ties will occur with probability zero.

10 For example, if there are two decisions, D_1 and D_2, with associated expected payoffs of π^e_1 and π^e_2, then the logit probability of choosing D_i is an increasing (exponential) function of its expected payoff:

$$Pr(D_i) = \frac{\exp(\pi^e_i/\mu)}{\exp(\pi^e_1/\mu) + \exp(\pi^e_2/\mu)} \qquad i = 1, 2,$$

where the denominator ensures that the probabilities sum to one.

equilibrium. By substituting the expected payoff from (1) into the logit choice density, we obtain:

$$f_i(b) \;=\; f_i(0)\,\exp\Big((V_i \,\Pi_{j\neq i} F_j(b) - cb)/\mu\Big), \quad b \in [0, B], \quad i = 1,\dots,n. \tag{2}$$

The density in (2) is greatest at the bid that yields the highest expected profit, but non-optimal bids have densities that are non-zero, and increasing in the expected payoffs for those bids. The parameter μ reflects the degree of irrationality: as μ tends to infinity, the density function in (2) becomes flat over its whole support and behavior becomes random. As the error rate becomes smaller, decisions with higher payoffs are chosen with increasingly higher probability. In the limit as the error rate goes to zero, only optimal decisions are made, and we will show in the next section that the logit equilibrium converges to a Nash equilibrium.

The logit equilibrium condition is that the distribution functions that determine expected payoffs in (1) correspond to the choice densities determined in (2). It follows from differentiation of (2) that the equilibrium densities satisfy the logit differential equations:

$$f_i' \;=\; f_i\,\pi_i^{e\prime}/\mu, \qquad i = 1,\dots,n, \tag{3}$$

where the primes denote derivatives and the *b* arguments of the functions have been suppressed. Equations (2) and (3) are used in section IV to establish some general properties of the logit equilibrium: existence, symmetry (for the symmetric model), uniqueness when all prize values are equal, and comparative statics effects. These proofs are somewhat technical, and it is instructive to begin by considering the special case of two bidders.

III. The All-Pay Auction with Two Players

In this section we derive closed-form solutions for the logit equilibrium, both for identical and asymmetric values. In the symmetric two-player case with $V_1 = V_2 = V$, we can drop the subscripts and express the expected payoff in (1) as: $VF(b) - cb$. (The supposition that identical values lead to identical equilibrium distributions is justified by Proposition 3 below.) Substituting the derivative of this payoff function into (3) yields:

$$f' = f(Vf - c)/\mu . \tag{4}$$

This equation can be solved explicitly by dividing both sides by $f(Vf - c)$, and rewriting the result to obtain:

$$\frac{f'}{f(Vf-c)} = \frac{1}{c}\left(\frac{Vf'}{Vf-c} - \frac{f'}{f}\right) = \frac{1}{\mu}, \tag{5}$$

which can be integrated to give:

$$f(b) = \frac{c}{V}\,\frac{1}{1 - K\exp(cb/\mu)}. \tag{6}$$

The constant of integration, K, is determined by equating the integral of the density over $[0, B]$ to one, yielding

$$K = \frac{\exp((V - cB)/\mu) - 1}{\exp(V/\mu) - 1}. \tag{7}$$

It is readily verified that the expression for K given in (7) ensures that the density $f(b)$ in (6) is positive on its support. If $cB = V$, then $K = 0$, and the density in (6) is constant on its support. Moreover, it is independent of μ, so that the Nash equilibrium and logit equilibrium coincide.[11] It follows from (6) and (7) that the equilibrium density is everywhere decreasing when $cB > V$. The intuition is as follows. Given that people make errors, they will sometimes bid above the prize value, and this reduces the profitability of high bids to other players. Since the equilibrium density reflects relative profitability, it is falling away from the most profitable bid of zero.

For games with a finite number of strategies, McKelvey and Palfrey (1995) show that the quantal response equilibrium converges to a Nash equilibrium as the error parameter, μ, goes to zero. This approach can be used to provide an alternative derivation of the mixed-strategy Nash

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[11] The mixed-strategy Nash density entails constant expected payoffs over the entire interval, which in turn lead to a flat logit density by (2). This is why mixed-Nash and logit equilibria coincide when the mixed-Nash equilibrium is uniform over the whole range of feasible choices (Lopez, 1995).

equilibrium bid density, which is uniform on [0, V/c].[12] Baye, Kovenock, and de Vries (1996) show this is the only Nash equilibrium.

In a Nash equilibrium, expected payoffs are zero for both players because the support of the mixed strategies includes zero. Thus the sum of expected payoffs is zero, and the rent is exactly dissipated in this mixed equilibrium, irrespective of whether the cost c is high or low. It is useful to disentangle two effects in this result: in a symmetric equilibrium, the probability of winning is 1/2, so the expected payoff is $V/2 - c\,\mathrm{E}\{b\}$, where $\mathrm{E}\{b\}$ denotes the expected bid. Since the expected payoff is zero in a Nash equilibrium, a reduction in the cost parameter is exactly offset by an increase in bids. The behavioral adjustment in a logit equilibrium is less extreme, which can lead to over-dissipation of rents. This can be shown by using (6) to calculate expected bids, and hence, net rents for specific values of c and μ. Figure 1 shows the relationship between c and the sum of the two players' net rents, $V - 2c\,\mathrm{E}\{b\}$, for the case of $V_1 = V_2 = 1$ and $B = 2$. As expected, the rent is exactly dissipated when $c = V/B = ½$ and is over-dissipated (negative net rents) for higher values of c.[13] The curve in Figure 1 with the lowest net rent corresponds to the highest error rate of 0.3. Notice that net rent is increased by lowering the bid cost, e.g., by making the waiting line more comfortable, in contrast with the Nash prediction.

We now consider the effect of value asymmetries, which can lead to unintuitive comparative statics in a Nash equilibrium. In particular, if $V_1 > V_2$, the Nash density for player 1 is independent of that player's value, V_1. To show this, note that the expected payoff for player 2 is $V_2 F_1(b) - cb$. In a mixed-strategy equilibrium, expected payoff is constant on its support,

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[12] There are two cases to consider, depending on whether cB is equal to or greater than V. When $cB = V$, the density in (6) is the uniform density, $f(b) = 1/B$ on [0, B]. Since this result is independent of μ, this is also the mixed-strategy Nash equilibrium. When $cB > V$, the term $K\exp(cb/\mu)$ in (6) converges to $-\exp((cb-V)/\mu)$ as μ goes to zero, so $f(b)$ in (6) converges to $(c/V)\,[1+\exp((cb-V)/\mu)]^{-1}$. As μ tends to zero, this density converges to c/V for $b < V/c$, and to zero for higher values of b. Thus when the cost parameter is high, the upper limit of the support is V/c, which is less than the maximum allowable bid, B, and the mixed-strategy equilibrium density is uniform on [0, V/c].

[13] Numerical calculations used to construct Figure 1 show that there is under-dissipation when c is below ½. Indeed, over-dissipation is impossible (for any μ) whenever $c < V/2B = ¼$ since then even maximal bids cannot dissipate the rent. Over-dissipation can occur when $c < V/B$ if there are more than two players and a sufficiently high error rate; see the discussion following Proposition 6.

so the equilibrium distribution F_1 is a function of V_2, but not V_1.[14] In contrast, the logit equilibrium has the property that an increase in the higher value, V_1, will stochastically increase player 1's bids. We will demonstrate this result by solving for the equilibrium densities in the asymmetric case.

With value asymmetries, the equilibrium conditions in (3) become:

$$\begin{aligned} f_1' &= f_1 \left(V_1 f_2 - c\right)/\mu, \\ f_2' &= f_2 \left(V_2 f_1 - c\right)/\mu. \end{aligned} \tag{8}$$

Multiply the top equation by V_2 and subtract the bottom equation multiplied by V_1, to obtain:

$$V_2 f_1' - V_1 f_2' = -\frac{c}{\mu}(V_2 f_1 - V_1 f_2),$$

which can be integrated to yield:

$$V_1 f_2(b) = V_2 f_1(b) + A \exp(-cb/\mu), \tag{9}$$

where A is a constant of integration. Since we have $V_1 f_2$ as a function of $V_2 f_1$, we can write the top equation in (8) as a differential equation in f_1 only. We found a closed-form solution that is similar to a double exponential distribution.[15] It is more interesting, however, to consider the

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[14] To be more precise, it can be shown that player 2's equilibrium expected payoff is zero, regardless of V_1. First, note that it cannot be the case that both players have spikes of probability at zero, since each player then would have an incentive to raise a bid a little above zero to get a finite increase in the probability of winning. In a mixed-strategy equilibrium, the expected payoff of player i is equated to a constant, E_i: $V_2 F_1(b) - cb = E_2$ and $V_1 F_2(b) - cb = E_1$. These equations imply that $F_1(b)$ will be steeper than $F_2(b)$, so the only possible configuration is that $F_1(0) = 0$, and $F_2(0) > 0$, i.e., player 2 is the one with a spike of probability at 0. Thus, $E_2 = 0$, and it follows that $F_1(b) = cb/V_2$, which is independent of V_1.

[15] The differential equation in f_1 is: $f_1' = f_1 (V_2 f_1 + A \exp(-cb/\mu) - c)/\mu$, and it can be verified by substitution that the solution is

$$f_1(b) = \frac{\exp(-(A/c)\, e^{c(B-b)/\mu} + c(B-b)/\mu)}{(V_2/A)\exp(-(A/c)\, e^{c(B-b)/\mu}) + K^*},$$

where K^* is a constant that forces the density to integrate to one. The solution for the other player's density is obtained by replacing V_1 by V_2. These formulas could be useful in evaluating data from laboratory experiments.

effects of changes in values on players' bid distributions.

Proposition 1. With two players, an increase in a player's value results in an increase in that player's bids (in the sense of first-degree stochastic dominance).

Proof. Without loss of generality, let player 1 be the one whose value will increase. Equation (9) can be integrated from 0 to b to obtain: $V_1 F_2(b) = V_2 F_1(b) + (\mu A/c)\,[1 - \exp(-cb/\mu)]$. Evaluating this equation at $b = B$ we can determine the value of the integration constant: $A = (c/\mu)\,(V_1 - V_2)\,[1 - \exp(-cB/\mu)]^{-1}$. Using these expressions, the expected payoff for player 1, $V_1 F_2 - cb$, can be written in terms of F_1, which allows us to write the density condition in (2):

$$f_1(b) \;=\; f_1(0)\,\exp\left(\frac{1}{\mu}\left(V_2 F_1 \;+\; (V_1 - V_2)\,\frac{1 - \exp(-cb/\mu)}{1 - \exp(-cB/\mu)} \;-\; cb\right)\right). \qquad (10)$$

This is the equation that determines the density of player 1 in the asymmetric model. When player 1's value is increased to V_1^*, the corresponding formula for the density, f_1^*, becomes

$$f_1^*(b) \;=\; f_1^*(0)\,\exp\left(\frac{1}{\mu}\left(V_2 F_1^* \;+\; (V_1^* - V_2)\,\frac{1 - \exp(-cb/\mu)}{1 - \exp(-cB/\mu)} \;-\; cb\right)\right). \qquad (11)$$

The structure of the proof is to show that there can only be two crossings of the distribution functions, F_1 and F_1^*, and since the distributions are equal at 0 and B, these are the only crossings. We subsequently show that F_1 starts out above F_1^* at low bids, so that bids are stochastically higher under F_1^*. At any crossing, $F_1 = F_1^*$, and it follows from (10) and (11) that the ratio of the slopes at a crossing is:

$$\frac{f_1^*(b)}{f_1(b)} \;=\; \frac{f_1^*(0)}{f_1(0)}\,\exp\left(\frac{1}{\mu}\,(V_1^* - V_1)\,\frac{1 - \exp(-cb/\mu)}{1 - \exp(-cB/\mu)}\right). \qquad (12)$$

The ratio on the right side of (12) is strictly increasing in b since $V_1^* > V_1$. If there were more than two crossings, the ratio of slopes at successive crossings would either decrease and then increase, or the reverse, a contradiction. Since the ratio in (12) is increasing, it must be less than one at $b = 0$ and greater than one at $b = B$. Therefore $F_1 > F_1^*$ for all interior values of b. The proof for the other player is analogous. Q.E.D.

In particular, notice that a value increase for the player with the higher value will stochastically raise that player's bids. As noted above, this intuitive result is not a property of the mixed-strategy Nash equilibrium. Similar results hold when players have identical prize values but different bid costs, c_i. In particular, if $c_1 > c_2$, then player 2's Nash equilibrium bid density is independent of c_2, whereas the logit equilibrium predicts that player 2 will bid stochastically less in response to a higher bid cost. The effects of cost and value asymmetries on rent dissipation are considered in section V below.

IV. The All-Pay Auction with n Players

Although closed-form solutions are not available for the general asymmetric n-player case, we can prove some existence, symmetry, and uniqueness results that are useful in the subsequent analysis of comparative statics effects and rent dissipation.

Proposition 2. A logit equilibrium exists for the n-player all-pay auction.

The general existence proof for the n-player, asymmetric-value case is given in Appendix A. The proof is based on Schauder's fixed-point theorem, which is a generalization of Brouwer's theorem to function spaces (which are not compact).[16] When all the players have the same prize value, we are to able to derive the closed-form solution, which is useful for the analysis of rent dissipation (see Appendix B).

The next issue is symmetry. The equilibrium will not be symmetric across players if their values differ. In particular, the player with the higher value will have stochastically higher bids. Nevertheless, we can show that those with identical values have the same bid distributions, even if others' values are different.

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[16] Other applications of Schauder's theorem are given in Stokey and Lucas (1989), Ch. 17.

Proposition 3. In any logit equilibrium for the all-pay auction, players with identical values have identical bid distributions. When players have different values, those with higher values bid more (in the sense of first-degree stochastic dominance).

Proof. We start by proving the final statement in the proposition. Let F_1 and F_2 denote the distributions corresponding to V_1 and V_2, where $V_1 > V_2$. Suppose that $F_1 = F_2$ on some interval of bids. Then the derivatives of these distributions must also be the same on this interval, which is impossible by considering (2). At any crossing of the distribution functions, $F_1 = F_2 = F$, and it follows from (2) that the ratio of the slopes at all such crossings is:

$$\frac{f_1(b)}{f_2(b)} = \frac{f_1(0)}{f_2(0)} \exp\left(\frac{1}{\mu}(V_1 - V_2)\, F \prod_{j\neq 1,2}^{n} F_j\right), \tag{13}$$

which is strictly increasing in F, and hence in b. If there were three crossings, the ratio of slopes at successive crossings would either decrease and then increase, or the reverse, a contradiction. Since the ratio in (13) is increasing, it must be less than one at $b = 0$ and greater than one at $b = B$. Therefore $F_2 > F_1$ for all interior values of b. Next, consider the case of equal values: $V_1 = V_2 = V$, for which we must show that the bid densities for players 1 and 2 are identical. Consider a particular value of b. Since $F_2(b) > F_1(b)$ for all $V_1 > V_2$, and $F_2(b) < F_1(b)$ for all $V_1 < V_2$, it follows from a continuity argument that the distributions are equal at b when $V_1 = V_2$. Obviously, this argument holds for all values of b. Q.E.D.

It is readily verified that when players have identical prize values but different bid costs, c_i, those with lower bid costs bid (stochastically) higher. The proposition implies that the equilibrium will be symmetric when all players' costs and values are identical. This symmetry result is interesting because Baye, Kovenock, and de Vries (1996) have shown that there can be asymmetric mixed-strategy Nash equilibria (with no errors), even in a symmetric model, as long as there are more than two players. The effect of errors is to "smooth out" the best response functions in a way that precludes asymmetric equilibria in the symmetric model.[17] In addition,

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[17] In the limit that μ goes to zero, the logit equilibrium "selects" the symmetric mixed-strategy equilibrium.

we can show that the symmetric equilibrium is unique.

Proposition 4. The logit equilibrium is unique when values are identical.

Proof. In light of the symmetry result in Proposition 3, it suffices to show that there is at most one symmetric equilibrium. Suppose in contradiction that there are two symmetric equilibria, distinguished by "I" and "II" subscripts. By dropping the player-specific subscripts from (3) and using the derivative of the payoff in (1), we obtain the following differential equations for the two candidate solutions:

$$f_I' = f_I \left(V f_I F_I^{n-2}(n-1) - c\right)/\mu,$$
$$f_{II}' = f_{II} \left(V f_{II} F_{II}^{n-2}(n-1) - c\right)/\mu. \qquad (14)$$

Without loss of generality, suppose that $f_I(b) = f_{II}(b)$ for b less than some bid b_L, possibly zero, and that $f_I(b) > f_{II}(b)$ for b just above b_L. Since the densities must integrate to one, they must cross again at some higher bid, b_U, with f_I crossing from above. Thus, at $b = b_U$, it must be the case that $f'_I \leq f'_{II}$. However, at b_U, we also have $f_I = f_{II}$ and $F_I > F_{II}$, which together with (14) implies $f'_I > f'_{II}$, a contradiction. Q.E.D.

It is more difficult to establish uniqueness of the logit equilibrium in asymmetric models, but the special case of two players is tractable. Recall that (10) determines the bid density for player 1 in the asymmetric model for $n = 2$. Suppose that there were two solutions, f_1 and f_1^*, to this equation. Taking the ratio of (10) to the analogous equation for f_1^*, we obtain:

$$\frac{f_1(b)}{f_1^*(b)} = \frac{f_1(0)}{f_1^*(0)} \exp\left(V_2 (F_1 - F_1^*)/\mu\right). \qquad (15)$$

If the initial conditions for the two candidate solutions were equal, then the differential equation in (10) would trace out the same density in each case. So without loss of generality, let $f_1(0) > f_1^*(0)$. Since both densities integrate to one, they must cross at some interior point, b_c. At the crossing, $f_1(b_c) = f_1^*(b_c)$ and $F_1(b_c) > F_1^*(b_c)$, which together with (15) contradict the initial assumption that $f_1(0) > f_1^*(0)$. Hence the bid distribution for player 1 is unique. An analogous argument establishes uniqueness for the other player.[18]

Next we consider the effects of changes in the exogenous parameters on the equilibrium bid distributions. As would be expected, bids are stochastically decreasing in the cost parameter, c, and stochastically increasing in both the value parameter, V, and the uppermost bid, B.

Proposition 5. In the logit equilibrium for a symmetric all-pay auction, bids are raised in the sense of first-degree stochastic dominance by a decrease in the cost parameter, c, an increase in the common value, V, or an increase in the maximum feasible bid, B.

Proof. Let c_I and c_{II} be the common cost parameter for all players, where $c_I > c_{II}$, and let F_I and F_{II} denote the corresponding distributions. As before, the structure of the proof is to show that the distribution functions can only cross twice, at the boundaries. At any crossing, $F_I = F_{II}$, and it follows from (2), applied to the symmetric case, that the ratio of the slopes at a crossing is:

$$\frac{f_{II}(b)}{f_I(b)} = \frac{f_{II}(0)}{f_I(0)} \exp((c_I - c_{II})b/\mu). \tag{16}$$

The right side of (16) is strictly increasing in b. By the argument used in Proposition 4, this implies that the only two crossings are at the boundaries, and $F_I > F_{II}$ for all interior values. The proof for an increase in the players' common value is analogous. The effect of an increase in the maximum feasible bid, B, is also proved along the same lines. As before let F_I, F_{II} denote the distribution functions corresponding to B_1 and B_2, where $B_1 > B_2$. For all b ε $[0, B_2]$, equation (2) implies that $f_I(b)/f_{II}(b) = f_I(0)/f_{II}(0)$ at any crossing of F_I and F_{II}, i.e. the ratio of slopes of the distribution functions at any crossing on $[0, B_2]$ must equal the ratio of their slopes at $b = 0$. Therefore, the distribution functions can only cross once on this interval, i.e. at $b = 0$. Since F_2 reaches its maximum value of 1 at B_2, it lies everywhere above F_1, which reaches its maximum value of 1 at $B_1 > B_2$. Q.E.D.

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[18] We have not been able to use this method to prove uniqueness in the case of value asymmetries with more than two players. Possible non-uniqueness would not affect any of the propositions. In particular, the result of Proposition 3, which pertains to the n-player asymmetric case, holds for *any* logit equilibrium.

In the benchmark case where $c = 1$ and $B = V$, it is natural to consider what happens when B and V are raised by the same amount. A simultaneous increase in B and V can be decomposed into an increase of V holding B constant, followed by an increase in B holding V constant. It follows from Proposition 5 that the combined effect will raise bids, since each effect alone raises bids. The method of proof for Proposition 5 cannot be used to determine the comparative static effect of the number of bidders, n, on the bid distributions. In the next section, we evaluate the effect of n on net rents, using the closed-form solution for the equilibrium distribution derived in Appendix B.

V. Rent Seeking

Allocations based on lobbying effort are likely to be used when ethical or equity considerations preclude selling the prize outright as in a market transaction.[19] For example, it is not reasonable to expect that the benefits that academic Deans dispense could be sold in this manner. Effort-based competitions, however, have the undesirable feature of using up real resources, and this rent dissipation can be considerable. In fact, when prize values are equal, rents are fully dissipated in a Nash equilibrium, and are over-dissipated for a wide range of parameter values in the presence of errors, as shown in Proposition 6 below. Here an equal division of the prize, if this is feasible, is more efficient since there is no unnecessary expenditure of real resources. (Many prizes such as grant money, funding for computers, or teaching reductions, are fairly divisible, despite the fact that they are usually allocated in discrete lumps.) The counter-argument, in favor of using effort-based competitions, is that contestants with higher values will exert more effort, and hence, have a higher probability of winning. This raises the issue of just how much value asymmetry is required for the all-pay auction to achieve a higher net rent than a simple random allocation or equal division. Before dealing with value asymmetries, we consider rent dissipation in a symmetric model.

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[19] In addition, the person awarding the prize may care about more than just the net value of the prize to the contestants. The person making the award may have an independent preference in favor of one of the contestants. This could make a significant difference if the rent seeking efforts are closely balanced. In other cases, the person making the award decision may enjoy the attention that comes with rent-seeking efforts. In this section, we restrict attention to the total net rents for the contestants. These qualifications should be kept in mind when we use the term over-dissipation.

The pattern of rent dissipation, shown in Figure 1 for the symmetric, two-player model, shows over-dissipation for high costs, i.e., when $c > V/B = 1/2$. We now show that this result is true more generally.

Proposition 6. In a logit equilibrium, there is over-dissipation of rent in the symmetric-value model with more than two players and $cB \geq V$.

Proof. Recall that the logit equilibrium corresponds to the Nash equilibrium in the limit as μ goes to zero. For the Nash mixed-strategy equilibrium, the lower bound of the support is zero, and therefore expected payoffs are zero. Hence, the expected net rent is zero, so there is exact dissipation in the absence of errors. In addition, the symmetric mixed-strategy Nash equilibrium bid density, $f_*(b) = (n\text{-}1)^{-1}(c/V)^{1/(n-1)}\ b^{(2-n)/(n-1)}$ is clearly decreasing in b when $n > 2$ (Baye, Kovenock, and de Vries, 1996). Dividing both sides of the formula for the logit equilibrium density in (2) by the Nash density $f_*(b)$ yields

$$\frac{f(b)}{f_*(b)} = \frac{f(0)}{f_*(b)} \exp\left([V F(b)^{n-1} - cb]/\mu\right). \qquad (17)$$

Recall that $VF_*^{n-1}(b) - cb = 0$ for all b by construction of the mixed-strategy Nash equilibrium, so we know that the term in square brackets in (17) is zero when the Nash distribution, F_*, crosses the logit distribution, F. It follows that $f(b)/f_*(b) = f(0)/f_*(b)$ at crossings. Since $f_*(b)$ is decreasing, the ratio on the left side of (17) is increasing at successive crossings. Thus this ratio has to be less than one at the lower bound and greater than one at the upper bound, and hence, $F(b) < F_*(b)$ at all interior points. It follows that the logit equilibrium distribution lies below the Nash mixed-strategy distribution that fully dissipates the rent. A lower distribution function implies higher expected bids, which in turn implies that rent is over-dissipated. Q.E.D.

Proposition 6 gives a sufficient condition for over-dissipation, which includes the benchmark case where $c = 1$ and $B = V$. However, over-dissipation can also occur for $cB < V$. To see this, recall that as the error parameter goes to infinity, bidding behavior becomes completely random, i.e., uniform on the interval [0, B]. Therefore, the expected bid converges to $B/2$ for each player, and the total effort cost converges to $ncB/2$. Thus the total effort cost can exceed the prize value V for sufficiently large values of n and μ, even when $cB < V$.

Davis and Reilly (1994) conducted a series of all-pay auction experiments with financially motivated human subjects. In their treatments, $c = 1$ and the maximum possible bid was not restricted to be less than the prize value, so $cB \geq V$. There were four bidders, so Proposition 6 implies that rent will be over-dissipated. They report that the social costs of rent-seeking activities consistently exceeded the prize value, so subjects lost money on average. This tendency for losses was handled by providing each subject with a relatively large initial cash balance.[20]

Many annoying requirements that are imposed in all-pay competitions can be understood as attempts to limit the number of contestants, where direct exclusion may be perceived as being unfair. Indirect exclusion is probably intended to make the task of comparing bids easier for the person awarding the prize, but the effect of such limits may be to reduce rent dissipation as well. We can use the closed-form solution for the equilibrium distribution in the n-player model, equation (21) in Appendix B, to determine the effects of changing the number of bidders on net rents. Figure 2 shows the relationship between net rents and n, for selected values of the error parameter, μ. Notice that numbers restrictions raise net rents, but that the effect is small when the error rate is low, and indeed, rent is fully dissipated in the Nash equilibrium case of $\mu = 0$ for all n.

Finally, we investigate the effects of asymmetries on rent dissipation, using an equal division or random allocation of the prize as a basis of comparison. For simplicity, we restrict attention to two players. We start with different values $V_1 > V_2$ and identical bid costs, $c = 1$. Recall that net efficiency in an all-pay auction depends on the tradeoff between the costs of rent seeking and the increased probability that the prize is awarded to the person who values it the most. This tradeoff is very simple to evaluate with full rationality ($\mu = 0$). In a mixed Nash

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[20] In these experiments, the worst errors (bidding above value) tended to occur in early rounds. We would expect error rates to decline over time in stationary environments, but not to disappear altogether. In fact, the error rates estimated by McKelvey and Palfrey (1995) for simple matrix games decline in successive periods of laboratory experiments, although some residual noise remains in most cases.

equilibrium with $V_1 > V_2$, Baye, Kovenock, and de Vries (1996) show that the expected payoff for the high-value player is V_1 - V_2, whereas the expected payoff of the low-value player is zero. Thus total net rent in the Nash equilibrium is V_1 - V_2. In comparison, an equal division or random allocation involves no effort cost and produces a net rent of $(V_1 + V_2)/2$. It follows from these observations that the all-pay auction with fully rational bidders is less efficient than random allocation unless V_1 - V_2 exceeds $(V_1 + V_2)/2$, or equivalently, $V_1 > 3V_2$. This three-to-one ratio seems like a relatively large value asymmetry, considering that the all-pay auction is likely to be replaced by a unilateral, dictatorial allocation when value asymmetries are so great as to be obvious to all concerned *ex ante*.

In the presence of decision error, we can calculate net rents using the formulas for the bid densities in footnote 15. Figure 3 shows the relationship between net rent and the high value, V_1, when the low value is set equal to 1. The solid lines show net rent for the logit equilibrium for $\mu = 0.1$ and $\mu = 0.3$. The line with short dashes shows net rent for the Nash equilibrium. For comparison, the net rent line for the equal-division allocation is shown as a line with long dashes, which is higher when $V_1 < 3$. The effect of adding errors is to further increase the inefficiency of the all-pay auction.

Another aspect of the inefficiency of the all-pay auction is apparent when different players have different bid costs, c_i. Suppose for illustration that player 1's value exceeds that of player 2, but that the latter's bid cost is much lower. The optimal allocation is to give the prize to player 1. In a Nash equilibrium, player 2 will bid more aggressively when $V_1/c_1 < V_2/c_2$, and therefore the player with the lower value will win more often. Cost differences are a major determinant of the final allocation in the auction, but irrelevant to the optimum. The all-pay auction is inefficient not only because of the rent dissipation, but also because it relies on a sorting criterion that is inappropriate for choosing the contestant who would benefit the most.

VI. Conclusion

The all-pay auction has been widely studied because it is an allocation mechanism in which competition for a prize involves the expenditure of real resources, e.g., lobbying. Since losers also incur costs, economists have considered the extent to which rents associated with the prize are dissipated by the competitive process. In theory, full dissipation is possible, but over-dissipation is impossible in a Nash equilibrium because a zero effort ensures a zero payoff. The approach taken here is to introduce the possibility that players are not perfectly rational: bid choices are probabilistic, with an error parameter that allows perfect rationality in the limit. The resulting *logit equilibrium* yields a continuous relationship between the extent of rent dissipation and the cost of bidding. The over-dissipation observed in the Davis and Reilly (1994) experiment is consistent with the predictions of the logit equilibrium.

The logit equilibrium provides a number of intuitive comparative statics predictions, which can be tested in laboratory experiments. In the symmetric-value case, for example, the extent of rent dissipation is increasing in the number of players, which might explain eligibility restrictions that are sometimes imposed. (In contrast, rent is fully dissipated in a Nash equilibrium.) With asymmetric values, the all-pay competition provides the high-value player with a higher probability of obtaining the prize, but the added cost of competitive efforts more than offsets this benefit unless value differences are relatively large. Value asymmetries suggest other cases in which the Nash and logit equilibria differ. With two bidders, the high-value player's bid distribution is independent of the player's own value in the Nash equilibrium (with no error). In contrast, increases in values result in stochastic increases in bids for the logit equilibrium.

Economists have long suspected that some of the most glaring inefficiencies in an economy arise in non-market allocations. The standard way of analyzing behavior in non-market situations is to apply the notion of a Nash equilibrium or some refinement thereof. Although mathematically appealing, game theory is difficult to evaluate empirically because the predictions often depend on subtle, difficult-to-measure effects of informational and preference asymmetries. To date, the most direct evaluations of game theory have come from the laboratory. Many experimental studies report systematic differences between Nash predictions and data patterns. Even where behavior appears to be converging to a Nash equilibrium, there is almost always some residual noise in the data. Statistical tests are generally based on adding symmetric noise to the Nash prediction (in an ad hoc manner), which becomes the baseline from which the significance of deviations can be assessed. The disparities between Nash and logit predictions in the all-pay auction, summarized above, indicate that modeling endogenous decision errors can be quite different from adding symmetric, exogenous noise to the Nash prediction. Moreover, the logit equilibrium is convenient for empirical work because it specifies a likelihood function, and because it nests the Nash equilibrium as a limiting case, i.e., it allows arbitrarily small deviations from perfect rationality, and provides a natural null hypothesis ($\mu = 0$). It is important to point out that the Nash equilibrium does provide reasonably good predictions in many contexts. The logit equilibrium should be viewed as a generalization of Nash that preserves its usefulness in organizing the data, but which offers a new perspective in explaining anomalies. Although the logit choice function can be derived from basic axioms, we recognize that it is a specific parameterization that can be generalized in a number of ways. Other parameterizations may generate even better predictions in specific contexts. Nevertheless, in terms of qualitative predictions, the logit equilibrium seems clearly better than the Nash equilibrium.

Appendix A: Proof of Proposition 2 (Existence of Equilibrium)

Let $\underline{F}(b)$ denote the vector of bid distributions, whose i th entry, $F_i(b)$, is the bid distribution of player i, for $i = 1,...,n$. Integrating the left and right-hand sides of (2), yields an operator T that maps a vector $\underline{F}(b)$ into a vector $\underline{TF}(b)$, with components:

$$TF_i(b) \;\equiv\; \frac{\int_0^b \exp\left([\, V_i \prod_{j\neq i} F_j(y) - c_i y \,]/\mu \right) dy}{\int_0^B \exp\left([\, V_i \prod_{j\neq i} F_j(y) - c_i y \,]/\mu \right) dy}. \tag{18}$$

The vector of logit equilibrium distributions is a fixed point of this operator, i.e., $TF_i(b) = F_i(b)$ for all $b \in [0, B]$ and $i = 1,...,n$. Since the right side of (18) is continuous in b even when the distributions F_j for are not, the equilibrium distributions are necessarily continuous. So there is no loss of generality in restricting attention to $C\,[0, B]$, the set of continuous functions on $[0, B]$. In particular, consider the set: $S \equiv \{F \in C\,[0, B] | \|F\| \leq 1\}$, where $\|\cdot\|$ denotes the sup norm. The set S, which includes all continuous cumulative distributions, is an infinite-dimensional unit ball, and is thus closed and convex. Hence, the n-fold (Cartesian) product $S^n = S \times ... \times S$, is a closed and convex subset of $C\,[0, B] \times ... \times C\,[0, B]$, the set of all continuous n-vector valued functions on $[0, B]$. This latter space is endowed with the norm $\|\underline{F}\|_n = \max_{i=1..n} \|F_i\|$. The operator T maps elements from S^n to itself, but since S^n is not compact, we cannot rely on Brouwer's fixed point theorem. Instead, we use the following fixed point theorem due to Schauder (see for instance Griffel, 1985):

Schauder's Second Theorem: If S^n is a closed convex subset of a normed space and H^n is relatively compact subset of S^n, then every continuous mapping of S^n to H^n has a fixed point.

To apply the theorem, we need to prove: (i) that $H^n \equiv \{\underline{TF} | \underline{F} \in S^n\}$ is relatively compact, and (ii) that T is a continuous mapping from S^n to H^n. The proof of (i) requires showing that elements of H^n are uniformly bounded and equicontinuous on $[0, B]$. From (18) it is clear that the mapping $TF_i(b)$ is non-decreasing. So $|TF_i(b)| \leq TF_i(B) = 1$ for all $x \in [0, B]$, $F_i \in S$, and $i = 1,...,n$, and elements of H^n are uniformly bounded. To prove equicontinuity of H^n, we must show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|TF_i(b_1) - TF_i(b_2)| < \varepsilon$ whenever $|b_1 - b_2| < \delta$, for all $F_i \in S$, $i = 1,...,n$. Consider the difference:

$$|TF_i(b_1) - TF_i(b_2)| = \frac{\left|\int_{b_1}^{b_2} \exp([V_i \prod_{j \neq i} F_j(b) - c_i b]/\mu)\, db\right|}{\int_0^B \exp([V_i \prod_{j \neq i} F_j(b) - c_i b]/\mu)\, db}.$$

We can bound the right side by replacing the distribution functions with 1 in the numerator and with 0 in the denominator, to obtain:

$$|TF_i(b_1) - TF_i(b_2)| \leq \frac{\left|\int_{b_1}^{b_2} \exp([V_i - c_i b]/\mu)\, db\right|}{\int_0^B \exp(-c_i b/\mu)\, db}.$$

This inequality is maintained if b is replaced by 0 in the integrand of the numerator and by B in the integrand of the denominator. Then integration yields:

$$|TF_i(b_1) - TF_i(b_2)| \leq \frac{|b_2 - b_1| \exp(V_i/\mu)}{B \exp(-c_i B/\mu)}.$$

Thus the difference in the values of TF_i is ensured to be less than ε for all $F_i \in S$, $i = 1,...,n$, by setting $|b_1 - b_2| < \delta$, where $\delta = \varepsilon B \min_{i=1..n} \exp((-V_i - c_i B)/\mu)$. Therefore, $\underline{TF}$ is equicontinuous for all $\underline{F} \in S^n$.

Finally, we prove continuity of T. The mapping T is continuous if for all $\underline{F}^1, \underline{F}^2 \in S^n$ and for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|\underline{TF}^1 - \underline{TF}^2\|_n < \varepsilon$ when $\|\underline{F}^1 - \underline{F}^2\|_n < \delta$. In order to get a bound on $\|\underline{TF}^1 - \underline{TF}^2\|_n$, let us write $F_i^1(b) = F_i^2(b) + h_i(b)$ with $-\delta < h_i(b) < \delta$ for all $b \in [0, B]$, $i = 1 .. n$. Using the upper-bound, we derive

$$\prod_{j \neq i} F_j^1(b) < \prod_{j \neq i} (F_j^2(b) + \delta).$$

The right-hand side can be bounded by first extracting $\prod_{j \neq i} F_j^2(b)$ from the product, and then replacing all the distributions functions by 1 in the remainder, to get:

$$\Pi_{j\neq i} F_j^1(b) \;<\; \Pi_{j\neq i} F_j^2(b) + (1+\delta)^{n-1} - 1\,.$$

Similarly, using the lower-bound we derive:

$$\Pi_{j\neq i} F_j^1(b) \;>\; \Pi_{j\neq i} F_j^2(b) - (1+\delta)^{n-1} + 1\,.$$

Consider the definition of $TF_i^1(b)$, for $i = 1,\dots,n$, given in (18):

$$TF_i^1(b) \;\equiv\; \frac{\int_0^b \exp\Big([\, V_i \,\Pi_{j\neq i} F_j^1(y) - c_i y \,]/\mu \Big)\, dy}{\int_0^B \exp\Big([\, V_i \,\Pi_{j\neq i} F_j^1(y) - c_i y \,]/\mu \Big)\, dy}\,.$$

We can use the upper-bound for the product in the numerator, and the lower-bound for the product in the denominator, to obtain:

$$TF_i^1(b) \;<\; \frac{\int_0^b \exp\Big([\, V_i \,\Pi_{j\neq i} F_j^2(y) - c_i y + V_i\,((1+\delta)^{n-1} - 1)\,]/\mu \Big)\, dy}{\int_0^B \exp\Big([\, V_i \,\Pi_{j\neq i} F_j^2(y) - c_i y - V_i\,((1+\delta)^{n-1} - 1)\,]/\mu \Big)\, dy} \;=\; K_i(\delta)\, TF_i^2(b)\,,$$

where the constant is given by $K_i(\delta) = \exp(2V_i\,((1+\delta)^{n-1} - 1)/\mu)$. The same approach can be used to show that $TF_i^1(b) > K(\delta)^{-1}\, TF_i^2(b)$. Thus we conclude that

$$K_i(\delta)^{-1}\, TF_i^2(b) \;<\; TF_i^1(b) \;<\; K_i(\delta)\, TF_i^2(b)\,, \qquad i = 1 \,..\, n\,.$$

Notice that $K_i(\delta)$ is strictly increasing for $\delta > 0$, with $K_i(0) = 1$, for all $i = 1 \,..\, n$. The final step is to obtain a bound on $\|\underline{TF}^2 - \underline{TF}^1\|_n = \max_{i=1..n} \|TF_i^1 - TF_i^2\|$. Suppose without loss of generality that the (maximum) supremum is attained for $i = 1$, $b = b^*$, at which $TF_1^1 > TF_1^2$. We also have $TF_1^1 < K_1(\delta)\, TF_1^2$ at b^*. So

$$\|\underline{TF}^1 - \underline{TF}^2\|_n \;=\; |TF_1^1(b^*) - TF_1^2(b^*)| \;<\; (K_1(\delta) - 1)\; |TF_1^2(b^*)| \;\leq\; K_1(\delta) - 1\,.$$

Since $K_1(\delta)$ is continuous and increasing in δ with $K_1(0) = 1$, there exists a $\delta^*(\varepsilon)$ such that the far right side of the above inequality is less than ε for all $0 < \delta < \delta^*(\varepsilon)$. Hence T is a continuous mapping from S^n to H^n. Q.E.D.

Appendix B: Closed-Form Solution for the Symmetric Case

Here we derive the closed-form solution for the symmetric-value case. In a symmetric situation, the equilibrium condition in (2) simplifies to:

$$f(b) \;=\; f(0)\, \exp\left((V F^{n-1}(b) - c\,b)/\mu\right).$$

Multiplying both sides by $\exp(-V F^{n-1}/\mu)$, one obtains

$$\exp\left(-V F^{n-1}(b)/\mu\right) f(b) \;=\; f(0)\exp(-cb/\mu).$$

We integrate both sides from 0 to *b*, with *x* denoting the variable of integration, to get

$$\int_0^b \exp\left(-V F^{n-1}(x)/\mu\right) f(x)\, dx \;=\; f(0)\int_0^b \exp(-cx/\mu)\,dx. \tag{19}$$

The left side of (19) can be rewritten using the transformation of variables: $y = F^{n-1}(x)$, while the right side can be directly integrated so that:

$$\frac{1}{n-1}\int_0^{F(b)^{n-1}} y^{\frac{2-n}{n-1}}\exp\left(-Vy/\mu\right)dy \;=\; \frac{\mu}{c} f(0)\left[1-\exp(-cb/\mu)\right]. \tag{20}$$

When $b = B$, the upper limit of the integral on the left side of (20) is one, and the resulting equation can be used to determine *f*(0), which yields:

$$\int_0^{F(b)^{n-1}} y^{\frac{2-n}{n-1}}\exp\left(-Vy/\mu\right)dy \;=\; \frac{1-\exp(-cb/\mu)}{1-\exp(-cB/\mu)}\int_0^1 y^{\frac{2-n}{n-1}}\exp\left(-Vy/\mu\right)dy.$$

A closed-form solution for *F*(*b*) can be obtained by using the definition of the incomplete Gamma function: $\Gamma(z,a) \equiv \int_0^z t^{a-1}\exp(-t)\,dt$ (see for instance Wolfram, 1996). We find

$$F(b)^{n-1} \;=\; \frac{\mu}{V}\,\Gamma^{(-1)}\left[\frac{1-\exp(-cb/\mu)}{1-\exp(-cB/\mu)}\,\Gamma\!\left(\frac{V}{\mu},\frac{1}{n-1}\right),\frac{1}{n-1}\right], \tag{21}$$

where $\Gamma^{(-1)}$ denotes the inverse of the incomplete Gamma function. The closed-form solution in equation (21) was used to plot Figure 2.

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